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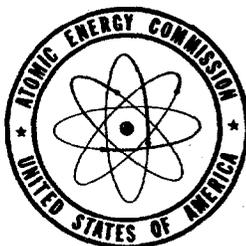
NEUTRON PHYSICS

A Revision of I. Halpern's Notes on E. Fermi's
Lectures in 1945

By
J. G. Beckerley

October 16, 1951
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Atomic Energy Commission
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III

PREFACE

In the Fall of 1945 a course in Neutron Physics was given by Professor Fermi as part of the program of the Los Alamos University. The course consisted of thirty lectures most of which were given by Fermi. In his absence R. F. Christy and E. Segre gave several lectures.

The present revision is based upon class notes prepared by I. Halpern with some assistance by B. T. Feld and issued first as document LADC 255 and later with wider circulation as MDDC 320.

Having found the document most useful in teaching an introductory course in nuclear physics, the author of the present revision felt that the material should be made more widely available, particularly to students of "pile engineering." To this end the notes issued as MDDC 320 have been revised and made available in this form for wider distribution.

The principal revisions in the text consist of expanding some of the statements for clarity and adding sentences and footnotes for completeness. Problems have been numbered and grouped at the end of each chapter. Figures have been redrawn, and in a few cases new ones added. Occasionally additional material has been included which may not have been presented in the lectures. This has been done only where clarity demanded more information and where the addition of recent data made the text more complete.

The reviser was not privileged to attend the course on which these notes are based. It is his hope, however, that the revision will make available to a wider group of students the essential material given in what must have been an extremely useful and informative course of lectures.

-J. G. B.

TABLE OF CONTENTS

I. NEUTRON SOURCES	
1.1 Alpha Neutron Sources	1
1.2 Photoneutron Sources	6
1.3 Neutron Sources Using Particle Accelerations	7
II. COLLISIONS OF NEUTRONS WITH NUCLEI	
2.1 General Types of Reactions	9
2.2 Neutron Cross Sections as a Function of Energy	9
2.3 Measurement of Total Cross Sections	11
2.4 The General Features of Collisions	13
2.5 Examples from Experiment	17
III. STABLE ISOTOPE CHART AND REACTIONS INVOLVING NEUTRONS	
3.1 The Segre Isotope Chart	23
3.2 Isotopic Weights and the Binding Energy of Neutrons	26
IV. MODELS OF NUCLEI AND NUCLEAR REACTIONS	
4.1 The Compound Nucleus	29
4.2 Neutron Resonances, Level Spacing	30
4.3 Two Nuclear Models	32
V. THE SCATTERING OF NEUTRONS	
5.1 The Breit-Wigner Formula	35
5.2 Some General Considerations on Neutron Scattering	36
5.3 Scattering by a Potential	37
5.4 The Scattering of Neutrons	43
VI. THE SLOWING DOWN OF NEUTRONS	
6.1 The Change of Direction and Energy upon Collision	50
6.2 Neutron Distribution from Point Source -- Experimental Methods	55
6.3 Distribution of Neutrons from Point Source -- Calculation of r_{av}^2	57
6.4 Distribution of Neutrons from Point Source -- Age Equation	62
VII. THE DISTRIBUTION OF SLOW NEUTRONS IN A MEDIUM	
7.1 The Differential Equation for Slow Neutrons	68
7.2 Boundary Conditions for the Slow Neutron Differential Equation	71
7.3 The Diffusion Length in Water and Graphite	73
7.4 The Albedo or the Reflectivity of Bounding Surfaces for Neutrons	76
VIII. NUCLEAR FISSION	
8.1 The Binding Energies of Nuclei	83
8.2 The Fission Process -- Energy Considerations	86
8.3 The Fission Process -- Resulting Particles	92
8.4 The Fission Process -- General Nature of Cross Sections	95

CHAPTER I
NEUTRON SOURCES

1.1 ALPHA-NEUTRON SOURCES

One useful type of neutron source is based on the (α, n) reaction. Alpha particles incident on a target nucleus result in the ejection of neutrons. Consider the reaction $\text{Li}^7(\alpha, n)\text{B}^{10}$:



In this equation B^{11} is the intermediate state or compound nucleus. Q is the reaction energy and can be calculated from the mass-spectrographically measured masses of the atoms as follows:

Li ⁷ :	7.01804 m. u.	B ¹⁰ :	10.01605 m. u.
He ⁴ :	4.00388	n ¹ :	1.00893
	11.02192 m. u.		11.02498 m. u.

$$Q = (\text{Mass on left}) - (\text{Mass on right}) = - .00306 \text{ m.u.}$$

The masses given are those of the neutral atoms, i.e., the "mass of an atom" is equal to the mass of the nucleus plus the mass of the associated electrons, and the units are defined by the relation

$$1 \text{ m. u.} = 1 \text{ mass unit} \\ = \left\{ \begin{array}{l} \text{Mass of a neutral atom of the most} \\ \text{abundant isotope of oxygen} \end{array} \right\} \div 16$$

Now, in equation (1-1) there is an excess of mass on the right side which means that the reaction is endothermic. That is, energy equivalent to the increased mass must be supplied to make the reaction energetically possible. This energy, denoted by Q , can be computed by conversion of the mass-difference $-.00306 \text{ m. u.}$, to energy units using the relation $E = mc^2$. Since 1 m.u. is equivalent to 931 Mev (see problem 2 at the end of this chapter) the value of the reaction energy for (1-1) is $Q = -.00306 \cdot 931 \text{ Mev}$ or -2.85 Mev . The *negative* sign indicates that the reaction is *endothermic*, i.e., Q is taken *positive* for *exothermic* reactions.

The "threshold energy" for this reaction is that *minimum value of the kinetic energy which the alpha must have in order to make the reaction energetically possible*. The threshold is not the same as the Q value, since the end-products, as a consequence of momentum conservation, will retain some kinetic energy. In the $\text{Li}^7(\alpha, n)$ reaction the incident alphas will need *more than* 2.85 Mev of kinetic energy.

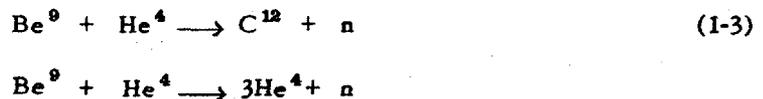
* A revision of class notes taken by I. Halpern on a series of lectures by Professor Enrico Fermi of the University of Chicago as explained in the preface to this document.

The exact threshold energy may be calculated by considering the initial momentum of the system. The total momentum of the colliding particles after the reaction will have to be the same as the total momentum before. Assuming the Li^7 to be initially stationary and the alpha particle to have a velocity v , then the total initial momentum of the system* is just $4v$. The velocity of the center of gravity v_c is this momentum divided by the total mass of the system or $4v/11$. The compound nucleus formed at collision (B^{11}) thus has a kinetic energy of $11v_c^2/2$, that is one half the total mass, 11 units, times the square of v_c . Substituting $4v/11$ for v_c yields $(4/11) \cdot 4v^2/2$ or just four-elevenths of the kinetic energy of the incident alpha particle. The balance or seven-elevenths of the kinetic energy of the incident alpha particle is thus available for nuclear excitation. For the reaction to be just possible this fraction of the alpha's kinetic energy must be just equal to the negative reaction energy Q :

$$\begin{aligned} -Q &= (7/11) (\text{Threshold K.E. of alphas}) & (1-2) \\ \therefore \text{Threshold K. E. of alphas} &= -(11/7)Q \\ &= -(11/7)(-2.85) \\ &= 4.48 \text{ Mev} \end{aligned}$$

Equation (1-2) means that in order for the reaction $\text{Li}^7 (\alpha, n) \text{B}^{10}$ to take place the incident alpha particles must have a kinetic energy of 4.48 Mev or greater.

An even more useful alpha-neutron reaction is that in which alphas are incident upon beryllium. The resulting reactions are *exothermic*:



in which the first reaction is more probable and takes place with $Q = +5.5$ Mev. Being exothermic there is no threshold. However, the Coulomb repulsion of the alpha particles by the beryllium nucleus diminishes the chance of a successful collision by a slow-moving alpha. The net result is that the yield of neutrons from a thin beryllium target (thin to reduce straggling effects from alpha-electron collisions) increases with increasing energy of the incident alpha particles as illustrated in the graph, Figure 1.

* The masses are simplified to the mass-numbers in this calculation, i.e., Li^7 , He^4 , B^{10} and n^1 are represented by masses 7, 4, 10, 1, respectively. This introduces a negligible error in the computed result.

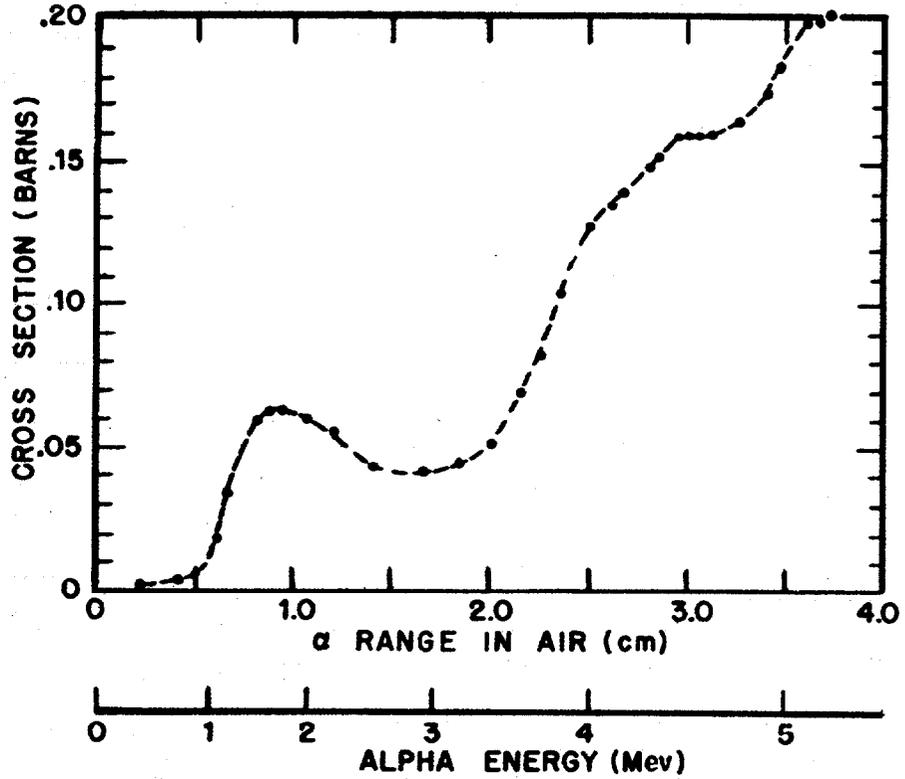


Figure 1. $\text{Be}^9 (\alpha, n)\text{C}^{12}$: Alphas on thin target of beryllium (0.22 mg/cm^2).
(I. Halpern, MDDC-716)

The reaction of alphas with beryllium is used as a neutron source. Alpha particles may be supplied by naturally radioactive substances such as radium, radon, and polonium. The characteristics of sources using these materials may be understood by examining the radium series, Figure 2.

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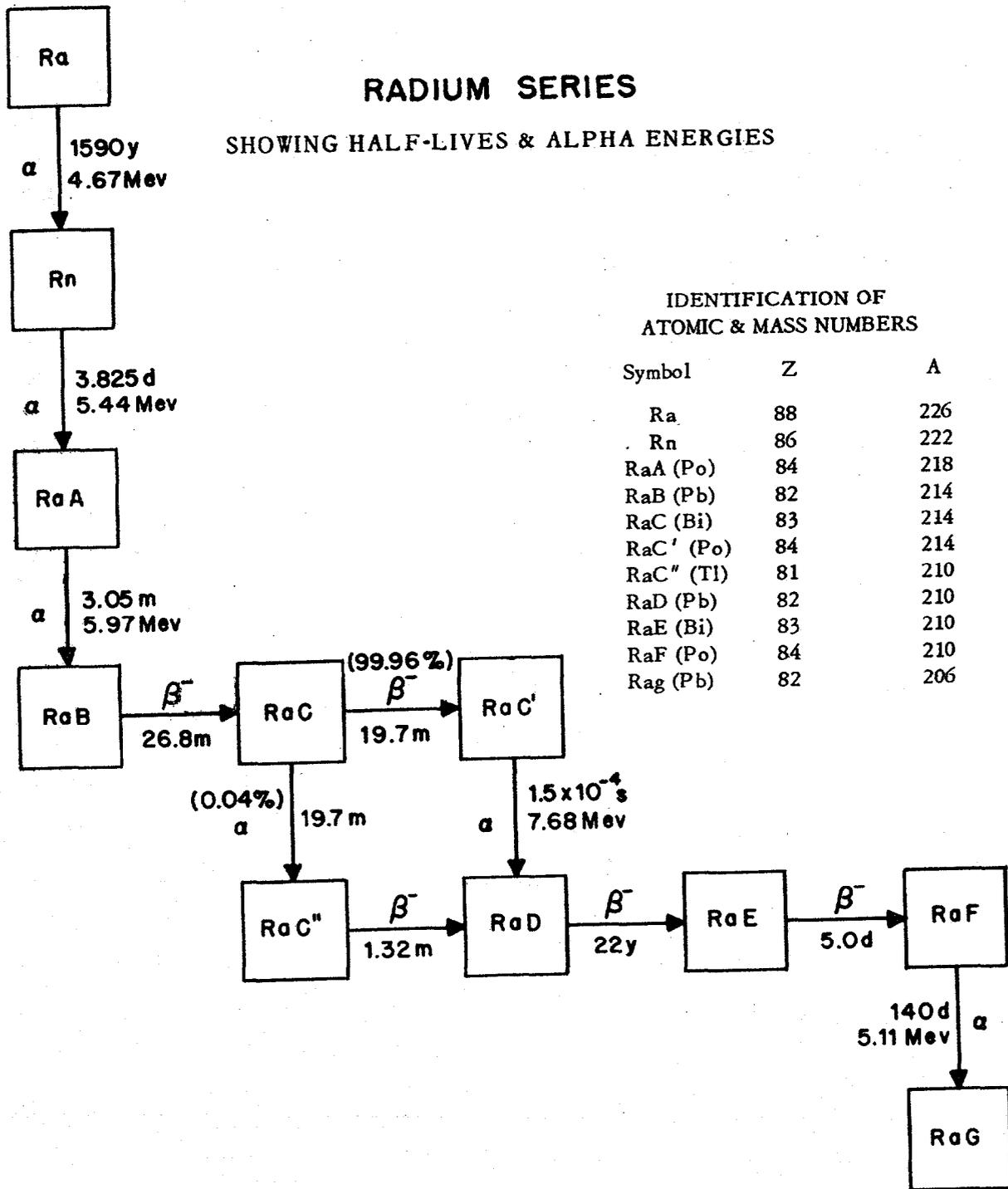


Figure 2

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The radium-beryllium neutron source has an advantage in its half-life period being long enough to make attenuation normally negligible during any experiment or series of experiments. The usual weight ratios of Be:Ra are from 5:1 to 3:1. A freshly prepared Ra-Be source must be "aged" to allow the daughter products (Rn, RaA, etc.) to come to equilibrium. The alpha-emitters among these daughter products contribute to the neutron production so that aging for about one month increases the neutron intensity by a factor of about 6 over the initial ("fresh" Ra) value.

As may be seen by inspecting the series, in radium aged for a month there will be alphas available from Ra, Rn, RaA, and Ra (C + C'); as a consequence the neutron spectrum will be complex* with neutron energies up to $7.68 + 5.5 \approx 13$ Mev. In addition the effect of passage of the alphas through the beryllium, even in finely powdered state, and the possibility of the resultant C^{12} nucleus being left in an excited state tends to make the Ra-Be source emit neutrons with a fairly continuous distribution of energies. A distinct limitation of the Ra-Be source is the accompanying gamma radiation.

This latter limitation is not present in the polonium-beryllium source. Polonium (RaF) emits alphas almost exclusively; the few gammas present in pure Po arise from the relatively improbable emission of alphas with energies slightly below normal.† Polonium-beryllium sources have a half-life of 140 days which limits their usefulness to some extent.

Radon can be used with beryllium. The gas is placed in a beryllium capsule. The yield is substantially the same as for a Ra-Be source; however, the half-life is only 3.8 days.

The strengths of these natural (α, n) sources will vary with the details of their construction. In round numbers Ra-Be and Rn-Be sources will emit about 1 to 2×10^7 neutrons/second/curie; Po-Be sources will emit about 2.8×10^6 neutrons/second/curie. Using the technique described by H. L. Anderson and B. T. Feld in the *Reviews of Scientific Instruments*, 18:186 (1947) a neutron yield is obtained for pressed Ra-Be sources as follows:

$$\text{Fast neutrons/second/gram Ra} = 1.7 \times 10^7 \frac{M_{\text{Be}}}{M_{\text{Be}} + M_{\text{Ra Br}_2}}$$

Absolute measurements with this type source have been reported by F. G. P. Seidl and S. P. Harris in the *Reviews of Scientific Instruments*, 18:897 (1947). Their "Source No. 38" consisting of 504 mc Ra and 3000 mg Be yielded $(5.5 \pm 0.4) \times 10^6$ neutrons/second. G. R. Gamertsfelder and M. Goldhaber in the *Physical Review*, 69:368 (1946) report a Ra-Be source yield of 6.8×10^3 neutrons/second/mc Ra.

Boron bombarded with alphas yields neutrons. The reactions are $B^{11}(\alpha, n)N^{14}$ and $B^{10}(\alpha, n)N^{13}$ with the former reaction predominant. Yields are $\sim 2 \times 10^6$ neutrons/sec/curie for boron-radium mixtures; BF_3 can be used with yields of $\sim 10^5$ neutrons/sec/curie.

Similarly, fluorine yields neutrons in the reaction $F^{19}(\alpha, n)Na^{22}$.

* A. I. Alichanow, *Comptes Rendus Academie Sciences, U.S.S.R.*, 20:429 (1939). W. Y. Chang, *Physical Review*, 70:632 (1946).

† W. Y. Chang in *Physical Review* 69:60 (1946) gives Po alpha spectra. H. T. Richards, L. Speck, I. H. Perlmann in *Physical Review* 70:118 (1946) discuss neutron spectra of Po-B and Po-Be sources.

1.2 PHOTONEUTRON SOURCES

Reactions of the (γ, n) type can be used for neutron production. The gamma radiation is produced naturally by radioactive or artificially radioactive sources. Targets are restricted to a few light elements, those elements in which a neutron is rather loosely bound. Beryllium (Be^9) and heavy hydrogen (H^2) are alone among the isotopes having low enough (γ, n) thresholds (1.63 Mev and 2.185 Mev respectively)* to be useful with natural gamma emitters.

The Ra- γ -Be source † yields neutrons in two energy groups (0.12 Mev and 0.51 Mev) since two radium gammas are above the threshold. A practical rule for calculating the total number of neutrons per second in a Ra- γ -Be source is: 1 gm of Ra at 1 cm from 1 gm of Be gives 3×10^4 neutrons/second.

A fairly complete survey of photoneutron sources has been made at Argonne National Laboratory. Various artificial radioactive gamma emitters have been used with beryllium and heavy water. For some of these the emitted neutron energies have been measured. Table 1 lists these data.

TABLE 1. PHOTONEUTRON SOURCES.

SOURCE	HALF-LIFE	NEUTRONS/SECOND/CURIE		NEUTRON ENERGY IN KEV		REFER ENCE †
		*STANDARD SOURCE	*OTHER SOURCE	MEAN	MAXIMUM	
$\text{NA}^{24} + \text{D}_2\text{O}$	14.8H	29.0×10^4	2.7×10^6	220	320	A, B
$\text{NA}^{24} + \text{BE}$	14.8H	14.0	2.4	800	1020	A, B
$\text{MN}^{56} + \text{D}_2\text{O}$	2.6H	0.31	0.029	220		A, C
$\text{MN}^{56} + \text{BE}$	2.6H	2.9	0.50	{ < 300 < 150	{ 375 < 150	A, B
$\text{GA}^{72} + \text{D}_2\text{O}$	14H	6.9	0.64	130		A, C
$\text{GA}^{72} + \text{BE}$	14H	5.9	1.04			A
$\text{IN}^{116} + \text{BE}$	54M	0.82	0.14	{ < 300 < 150	{ 400 < 150	A, B
$\text{SB}^{124} + \text{BE}$	60D	19.0	3.2	35	68	A, B
$\text{LA}^{140} + \text{D}_2\text{O}$	40H	0.68	0.062	130		A, C
$\text{LA}^{140} + \text{BE}$	40H	0.23	0.041	620		A, C
$\text{Y} + \text{BE}$	100D			220 ± 20		D

**Standard* source is one curie at a distance of one centimeter from one gram of target material.
Other source is described in Figure 1 of reference A.

†References: A. B. Russell, D. Sachs, A. Wattenberg, R. Fields, *Phys. Rev.*, **73**: 545(1948), on neutron yields.

B. D. J. Hughes, C. Egger, *Phys. Rev.*, **72**: 902(1947), on neutron energies.

C. A. Wattenberg, *Phys. Rev.*, **71**: 497(1947), on neutron energies.

D. R. D. O'Neal, *Phys. Rev.*, **70**: 1(1946), on neutron energies.

*M. L. Wiedenbeck and C. J. Marhofer, *Physical Review* **67**: 54(1945).

†G. R. Gamertsfelder and M. Goldhaber in *Physical Review* **69**: 36B(1946) report a Ra- γ -Be source with a yield of 62 ± 7 neutrons/second/Mc Ra.

Other short-lived photoneutron sources have been investigated (reference C of Table 1). These include F^{20} (12s) + Be, Al^{28} (2.4 m) + Be and Cl^{38} (37m) + Be, all of which have greater than 0.1 useful gamma ray per disintegration. (This is to be compared to Na^{24} with one gamma ray of 2.8 Mev per disintegration.) As^{76} (26.8h) + Be and As^{76} (26.8h) + D_2O provide relatively less efficient neutron sources with 0.1 and less than 0.01 useful gamma per disintegration, respectively.

In addition to using gamma radiation from specific radioisotopes the radiation from fission products may be used. Photoneutron yields from U^{235} fission products irradiating heavy water are described by S. Bernstein, W. M. Preston, G. Wolfe, R. E. Slattery in the Physical Review 71: 573 (1947) and also 72: 163 (1947).

Gamma radiation produced in betatron or Van de Graaf accelerators may be used to generate photoneutrons. Yield curves for Van de Graaf gammas on beryllium have been determined by M. L. Wiedenbeck, Phys. Rev. 69: 235 (1946).

1.3 NEUTRON SOURCES USING PARTICLE ACCELERATORS

The deuteron-deuteron reaction, $H^2(H^2, n)He^3$, can be used to produce neutrons. Deuterons, accelerated with any suitable source of electrostatic potential (e.g., Van de Graaf, Cockcroft-Walton, etc.), bombard a heavy ice or heavy paraffin target. Protons are produced at the same time by the reaction $H^2(H^2, p)H^3$, with approximately as many protons produced as neutrons. The $H^2(H^2, n)$ reaction being exothermic with $Q \sim +3.2$ Mev. accounts for fairly good yields at relatively low energies, Figure 3, since it is only necessary for the incident deuteron to penetrate the Coulomb barrier of the target deuteron, no extra energy for excitation being needed. Although the reaction has considerable advantage in yielding a reasonable number of monoenergetic neutrons for relatively low energies the practical impossibility of designing a suitable target limits the use of the reaction as a neutron source. (It must be remembered that practically all the deuterons are not successful in producing neutrons but, rather, generate heat in the target.)

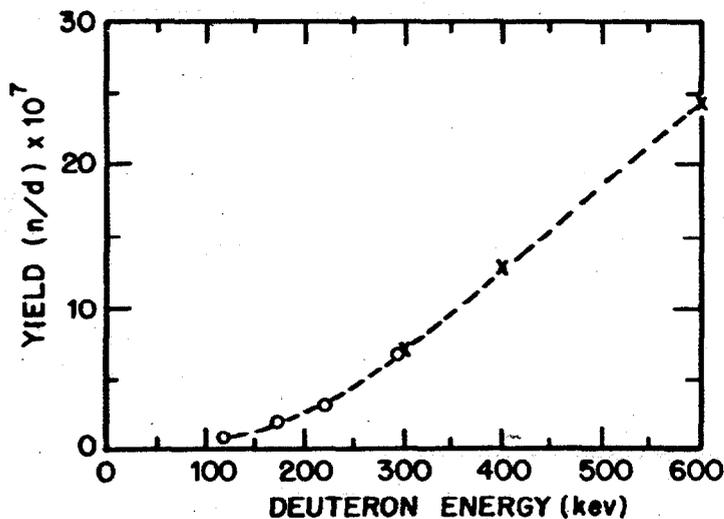


Figure 3

Another reaction using protons on lithium gives monoenergetic neutrons down to rather low energies. The reaction is $\text{Li}^7(\text{H}^1, \text{n})\text{Be}^7$ with a Q-value of -1.62 Mev. Similar to the example discussed in Section 1.1 the threshold is greater than Q, being in this case $8/7$ of 1.62 Mev or 1.85 Mev. If the lithium target is bombarded with protons of threshold energy the neutrons come off with finite energy, about 30 kev, for then they move with the speed of the center of gravity. If the proton energy is increased then there will be sufficient energy to give the neutrons a velocity *with respect to the center of gravity*. The net velocity is calculated by vector addition of the velocity of the center of gravity and the neutron velocity relative to the center of gravity, so that for high enough energies, neutrons can have resultant velocities of zero or even backward velocities. For a given proton energy, the energy of the neutrons will vary with the angle between the incident proton beam and the resultant neutron direction, that is for each angle of emergence of neutrons there will be a corresponding neutron energy. This will be discussed in greater detail in a later chapter.

Perhaps the most common neutron source is the "cyclotron source" in which deuterons bombard a beryllium target. In the reaction, $\text{Be}^9(\text{H}^2, \text{n})\text{B}^{10}$, the target is stable and can be made to dissipate the heat generated in the "non-successful" collisions. For a thick target 1 Mev deuterons give about 10^8 neutrons/sec/microamp; 8 Mev deuterons give 10^{10} neutrons/sec/microamp.

PROBLEMS

1. In the calculation of Q for the reaction of equation (1-1) explain why you can use atomic masses for such calculations, when it is true that nuclear masses alone are involved in the reaction.
2. Prove that one mass unit equals 931 Mev. Convert from mass units to grams; then, using $E = mc^2$, convert grams to ergs to Mev.
3. Given a hollow sphere of beryllium, inside radius 1 cm, outside radius 3 cm. A one gram capsule of radium is placed at the center of the sphere. Neglecting the absorption of the gamma radiation what is the approximate strength of this (γ, n) source? If one curie of Na^{24} were used in place of the radium, what would be the source strength?
4. How many neutrons per second would one expect from a heavy hydrogen target bombarded by one milliamper of 500 -kilovolt deuterons?

CHAPTER II

COLLISIONS OF NEUTRONS WITH NUCLEI

2.1 GENERAL TYPES OF REACTIONS

One of the most important types of collision processes is the "scattering" process. A scattering process is characterized by the identity of one of the ejected particles with the incident particle. If a neutron collides with a target nucleus (in a later chapter we shall define in detail what constitutes a "collision") and after the collision a neutron is observed leaving the scene, then it may be said that the neutron has been *scattered* by the target nucleus. If the kinetic energy of the neutron before the collision is equal to the sum of the kinetic energies of the recoil nucleus and scattered neutron after such a collision then the process is called "elastic scattering." If the kinetic energy is not conserved, i.e., some energy goes into nuclear excitation, then the process is "inelastic scattering." Using the customary notation (see equation 1-1), these definitions may be summarized as follows:

Scattering process:	$A (n,n)A^*$	
Elastic scattering:	$A = A^*$	(2-1)
Inelastic scattering:	$A^* = \text{excited state of } A$	

(A = target nucleus; A* = recoil nucleus)

For simplicity scattering processes are generally referred to as (n,n) processes.

In addition to the scattering process, collisions of neutrons with nuclei may result in the ejection of other particles or radiation, for example (n, γ), (n,p), or (n, α) reactions. The distinction between excited or stable resultant nucleus is not generally made.

There are other types of reactions occurring when neutrons collide with nuclei the most important of which are the (n,2n) and (n, fission) reactions. In the (n,2n) reaction a neutron incident on a nucleus results in the ejection of two neutrons, the recoil nucleus being isotopic with the target nucleus. This reaction is always endothermic. The (n, fission) or (n,f) reaction will be discussed at length in a later chapter. In the (n,f) reaction a neutron collides with a nucleus and as a result fission occurs.

2.2 NEUTRON CROSS SECTIONS AS A FUNCTION OF ENERGY

The collision of neutrons with nuclei may be described in terms of the target presented by the nuclei to the incident neutrons. Target size is specified by "cross section" or so many square centimeters per atom. It has become the custom to express cross sections in "barns" with one barn defined as 10^{-24} cm²/atom. If a beam of neutrons of density n neutrons per cubic centimeter all moving with a velocity v centimeters per second is incident on a nuclear target of area σ (cross section), then the number of neutrons hitting the target per second will be $n v \sigma$. This can be visualized by considering the target area σ to move with a velocity v through the neutron beam. In its motion the target will "sweep" out per second a volume $v \sigma$ in which there will be $n v \sigma$ neutrons, since there are n neutrons per unit volume.

While this representation is convenient it must be understood that the nucleus is not an area in the usual geometrical sense. Among other things the area is dependent on the energy of the incident particles. For neutrons this dependence may be understood qualitatively by considering the variation of the de Broglie wavelength of neutrons with energy. The fundamental relationship between the wavelength λ of a particle and its momentum p is $\lambda = h/p$ where h is Planck's constant. In terms of neutron energy this becomes:

$$E = mv^2/2 = (mv)^2/(2m) = p^2/2m \quad \text{or} \quad p = \sqrt{2mE} \quad (2-2)$$

$$\text{Thus } \lambda = h/p = h/\sqrt{2mE} \quad \text{for } v \ll c$$

The condition that v must be less than c , the velocity of light, is necessary since the non-relativistic expression for the kinetic energy has been used. (This limitation is negligible since the rest mass of the neutron is 931 Mev.) Expressing the neutron energy E in electron-volts and substituting $h = 6.61 \times 10^{-27}$ erg-sec., $m = 1.675 \times 10^{-24}$ gm, 1 electron-volt = 1.601×10^{-12} erg in the equation (2-2) yields the following convenient relation:

$$\lambda = \frac{0.286}{\sqrt{E}} \times 10^{-8} \text{ cm} \quad (E \text{ in electron-volts}) \quad (2-3)$$

Now, the distance between atoms in a solid is of the order of 3 or 4 Angstroms, that is $\sim 3 \times 10^{-8}$ cm. Putting numerical values into equation (2-3) shows that neutrons of about 0.01 ev have a characteristic wavelength of approximately interatomic dimensions, or very many times larger than any nuclear dimensions. To have a wavelength equal to the nuclear diameter of a medium weight nucleus, say 10^{-12} cm, the neutron energy would have to be about 10 Mev.

Now, a nucleus of diameter 10^{-12} cm, the diameter being defined as twice the range of nuclear forces, will have a "geometrical" cross section of about 10^{-24} cm² or 1 barn. If the neutron were a point particle then it would be reasonable to anticipate that all cross sections should be of the order of barns. However, as shown in equation (2-3) if the neutron behaves in a manner consistent with the fundamental basis of wave mechanics then a neutron only can be considered a "point" particle with respect to the nucleus when its wavelength is at least less than nuclear dimensions. This latter will be true only for fast neutrons, that is, with energies at least greater than 1 Mev. For slower neutrons the wavelength increases so that for thermal neutrons of 1/40 ev energy λ is about 2×10^{-8} cm. In this case it is the nucleus which is the "point" particle relative to the neutron so that the neutron size might be expected to determine the cross section. Thus σ should be of the order of λ^2 for slow neutrons. For thermal neutrons one might expect cross sections as large as 10^{-16} cm² or 10^8 times the fast neutron cross sections. Actually more rigorous theoretical analysis shows that λ^2 is an outside figure and that $\sigma < \lambda^2$.

The foregoing discussion is intended to present one argument showing that it is not possible to assign a unique geometrical cross section to a nucleus valid for all energies of incident neutrons, since in the range of energies generally considered in practical cases, say from 10^{-3} to 10^7 electron-volts, the neutron's "size" varies from 10^{-8} to 10^{-12} cm. Factors other than neutron size will be found to affect the cross section; these will be discussed in a later chapter.

In keeping with the idea that the cross section is not a strictly geometrical quantity the relation stated in the first paragraph of this section should be solved for σ :

$$\sigma = \frac{\text{Processes per nucleus per unit time}}{nv} \quad (2-4)$$

n = neutrons per unit volume in incident beam

v = neutron velocity

The quantity $n\nu$ is usually called the "neutron flux" since it is the number of neutrons incident per unit area per unit time on the target. Values of σ will obey the approximate inequality

$$10^{-24} \text{ cm}^2 \lesssim \sigma < \lambda^2 \quad (2-5)$$

where the de Broglie wavelength λ of the neutron is defined in equations (2-2) and (2-3).

2.3 MEASUREMENT OF TOTAL CROSS SECTIONS

The "total" cross section of a given material for incident neutrons is determined by measuring the neutron transmission of a known sample of the material. A source emits neutrons so that a beam of intensity I_0 (neutrons/sec/cm²) is incident on the sample, see Figure 4. As a result of scattering and absorption processes in passing through the material the neutron intensity is reduced to I . The detector response with the sample "in" the beam and "out" of the beam are measured so that I and I_0 , respectively, can be determined.

The geometrical arrangement must be such that any scattered neutron will not be detected. This means that the solid angle of the absorber at the source and at the detector must be very small. The broken line in Figure 4 shows how a scattered neutron might be detected if the neutron beam were not properly collimated. By defining the beam (i.e., making incident and transmitted beams parallel) with suitable apertures the geometrical conditions mentioned can be met. In addition to these geometrical conditions it is also necessary that any multiple scattering process in which a scattered neutron could be rescattered back into the beam has a negligible probability. This cannot always be satisfied for "thick" samples in which scattering predominates over absorption. (In the discussion of neutron diffusion and slowing-down, in a later chapter, it will be obvious that the exponential law to be derived herewith will not be valid.)

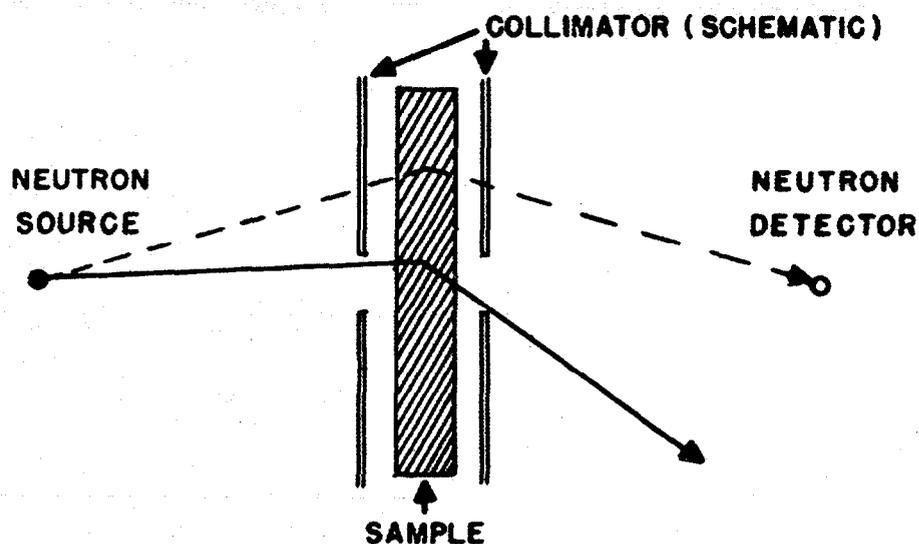


Figure 4. General arrangement for neutron transmission experiment (total cross section measurement).

2.4 THE GENERAL FEATURES OF COLLISIONS *

In any collision process the "initial state" consists of a particle incident on a target nucleus and the "final state" consists of an ejected particle and the recoil nucleus. For a given energy of the initial state there are a number of possible energies of the ejected particle as well as a number of possible energies of the recoil nucleus, provided these energies are consistent with energy conservation.

If the magnitude of the momentum of the ejected particle and its direction of motion are fixed, then momentum conservation fixes the momentum of the recoil and thereby the recoil's kinetic energy. The conservation of energy law then fixes the state of excitation of the recoil nucleus. Thus specifying the momentum (magnitude and direction) of the ejected particle specifies the final state completely.

Suppose the ejected particle has an energy E . Then what is the probability that a transition between initial and final states will occur? Assuming such transitions obey the general laws of quantum mechanics, it can be shown that the probability that a transition will occur in which the energy of the ejected particle is between E and $E + dE$ can be written as a product $M\rho$ where $M = |H|^2/K^2$ and $\rho(E)$ is the density of possible final states in the neighborhood of E . H is a matrix element which will be discussed qualitatively herewith and in a later chapter; π is Planck's constant divided by 2π .

The density function can be derived from statistical mechanics considerations. In a number of instances the variation of this factor will be found to be more effective than the variation of M . Using a familiar technique in quantum mechanics, the ejected particle is imagined to be in large box of volume Ω . This volume will be infinite in any practical case. The number of states of the ejected particle which will have an energy E in this box is proportional to the volume in phase space corresponding to this energy. With Cartesian coordinates where h is the linear dimension of a cell in phase space (a cell can contain one state), then the number of states for which x is between x and $x + dx$, y between y and $y + dy$, etc. and p_x is between p_x and $p_x + dp_x$, p_y between p_y and $p_y + dp_y$, etc. is just $dx dy dz dp_x dp_y dp_z / h^3$. Integrating over configuration space (x, y, z) reduces this to $\Omega dp_x dp_y dp_z / h^3$ or (Ω/h^3) times the momentum volume element. It is apparent then that the number of states for which the total momentum $p = \sqrt{p_x^2 + p_y^2 + p_z^2}$ is between p and $p + dp$ is just Ω/h^3 times the volume element between p and $p + dp$. This volume element, a spherical shell in momentum space, is $4\pi p^2 dp$. Thus

$$\begin{aligned} \text{Number of states with momentum between } p \text{ and } p + dp \\ = dN = (\Omega/h^3) 4\pi p^2 dp \end{aligned} \quad (2-9)$$

The density of states per unit energy range, $\rho(E)$, can now be calculated from (2-9) by changing from momentum to energy variables. For particles (α, p, n , etc.) $E = mv^2/2 = p^2/2m$ so that $dE = p dp/m = v dp$. For photons $E = h\nu = hc/\lambda$ or since $\lambda = h/p$ this means $E = pc$ or $dE = c dp$. (In both cases dE/dp is the particle or photon velocity.) Substituting these into (2-9):

$$\begin{aligned} \text{Number of states per unit energy interval with energy} \\ \text{between } E \text{ and } E + dE \\ \rho(E) = \frac{dN}{dE} = \frac{4\pi\Omega}{h^3} \cdot \frac{p^2}{v} \end{aligned} \quad (2-10)$$

$$\begin{aligned} \text{where } p^2/v &= m^2 v \text{ for particles} \\ &= h^2 \nu^2 / c^3 \text{ for photons} \end{aligned}$$

*In all of the discussion it is assumed that the ejected particle is in its "ground state," i.e., not excited.

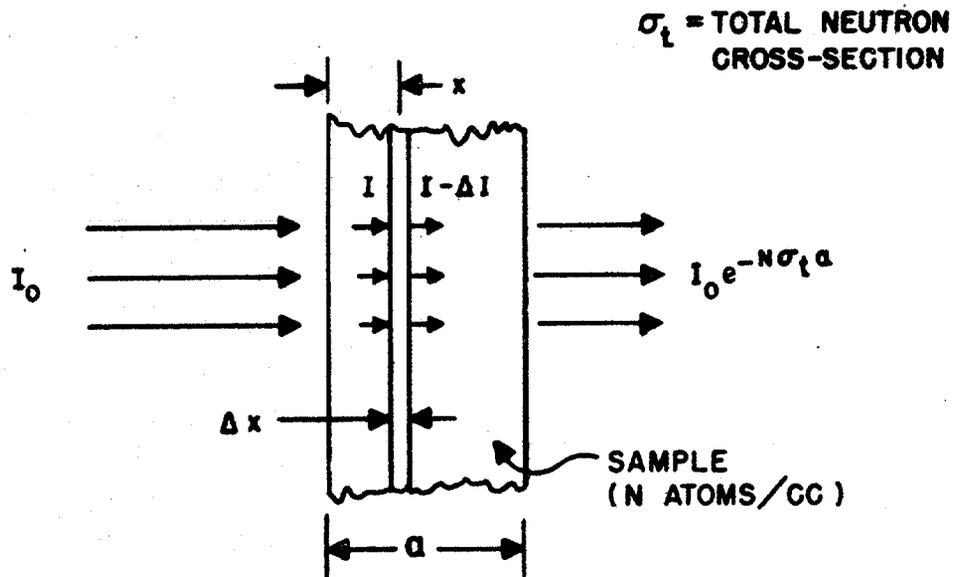


Figure 5. Exponential absorption law.

It is apparent that in the absorber (Figure 5) a layer of thickness Δx has a parallel beam of I neutrons/sec/cm² incident on it. If N is the number of absorber atoms per cubic centimeter then $N \Delta x$ nuclei per square centimeter are presented as "targets" for the incident neutron beam. From equation (2-4) it is apparent that the number of processes which will occur per unit area per unit time is $\sigma_t nv$ times the number of nuclei per unit area $N \Delta x$. Since nv is just the incident neutron flux I this means that

$$\text{Number of neutron collisions per unit area per unit time in layer } \Delta x = \sigma_t I N \Delta x \quad (2-6)$$

where $\sigma_t = \sigma_{\text{total}}$ = cross section for all collision processes

Now, any collision removes the neutron from the parallel beam so the above is just $-\Delta I$, the decrease in beam intensity. Equating and solving yields the familiar exponential law:

$$-\Delta I = \sigma_t I N \Delta x$$

$$-\Delta I/I = N\sigma_t \Delta x \quad (2-7)$$

$$I = I_0 e^{-N\sigma_t a} \quad (\text{integrating})$$

$$\text{or } \sigma_t = \frac{\log_e(I_0/I)}{N a}$$

where I_0 is the incident beam intensity (at $x = 0$) and a is the thickness of the sample. The last equation expresses the total cross section in terms of experimentally measurable quantities. The total cross section σ_t may be defined as

$$\sigma_t = \sigma_{\text{elastic}} + \sigma_{\text{inelastic}} + \sigma_{\text{absorption}} \quad (2-8)$$

where $\sigma_{\text{absorption}}$ includes all processes in which a neutron disappears, i.e., (n, γ) , (n, α) , (n, p) , etc.

The occurrence of Ω in $\rho(E)$ shows that as the volume, in which the ejected particle is confined, is allowed to become infinitely large the density of states becomes infinite or the particle can be ejected with any energy (i.e., a continuum of possible energy states). The transition probability does not become infinite, however, for the matrix elements contain Ω in such a way that M is proportional to Ω^{-1} . Thus the probability of a transition to a state where the ejected particle or photon has a momentum p_e (subscript for ejected or outgoing particle after the collision) is proportional to $M \cdot p_e^2/v_e$. Then for a single incident particle with velocity v_1 colliding with a single nucleus it follows from equation (2-4) that the probability of a collision (per unit time) is $v_1 \sigma$ where σ is the cross section for incident particles of velocity v_1 to result in outgoing particles of velocity v_e . Equating the probabilities and incorporating the various constants into M we obtain an equation for the cross section:

$$\sigma = M' \cdot \frac{p_e^2}{v_e} \frac{1}{v_1} \quad (2-11)$$

Equation (2-11) is too general for one to understand fully its significance. Let us apply the result to several specific situations:

Elastic Scattering

In an elastic scattering collision initial and final velocities are equal, i.e., $v_1 = v_e$. Substituting this into (2-11) reduces the cross section to

$$\sigma = M' m^2$$

This means the cross section is proportional to M' . For slow neutrons, where the energy range is small, the elastic scattering cross section will not depend appreciably on the neutron energy. It should be pointed out that for elastic scattering from nuclei of small mass number initial and final velocities are equal only in the center of gravity system and that it is in this system that the foregoing derivation is true.

Absorption Processes

Suppose the recoil is heavy so that the ejected (small mass) particle has almost all the kinetic energy in the final state. The situation is shown in Figure 6. A neutron of mass m , velocity v_1 hits the target nucleus, is absorbed and a new light (small mass) particle is ejected with mass m' and velocity v_e . Conservation of energy requires that the kinetic energy of the ejected particle must be equal to the kinetic energy of the incident neutron plus whatever energy Q is available from the nuclear reaction:

$$m' v_e^2/2 = m v_1^2/2 + Q$$

$$\text{or} \quad p_e^2 = (m')^2 \left[(m/m') v_1^2 + (2/m') Q \right]$$

Noting that $p_e^2/v_e = m' p_e$ and substituting in (2-11) gives the following expression for the cross section:

$$\sigma = M' (m')^2 \frac{1}{v_1} \left[(m/m') v_1^2 + (2/m') Q \right]^{1/2}$$

When Q is positive (exothermic reaction) and v_1 is small the cross section is proportional to $1/v_1$ — the so-called $1/v$ law. A negative Q and a very small v_1 does not make physical sense, giving an imaginary cross section and showing that the formula does not cover the case. σ should be zero for such a situation.

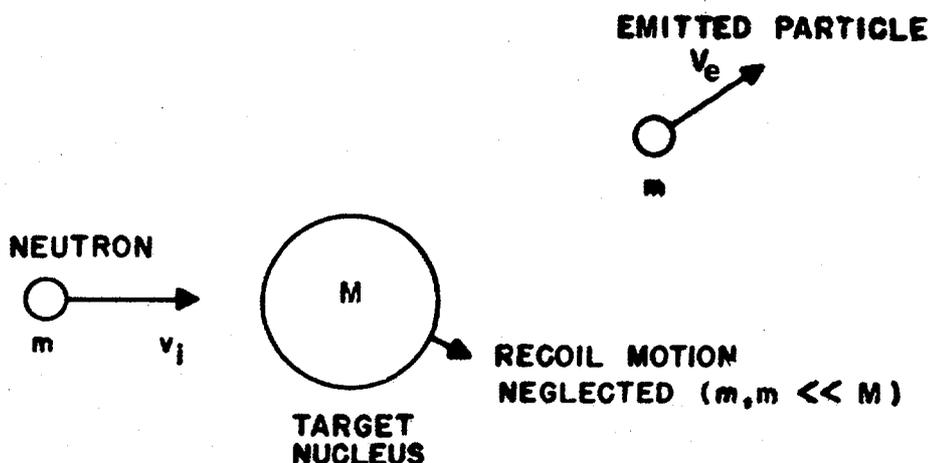


Figure 6. Absorption process.

Inelastic Scattering

Consider a neutron of mass m incident on a nucleus of mass M . If the target nucleus is not considered to be very heavy compared to the neutron then all variables should be referred to the center of gravity system. In this case the neutron mass must be replaced by its "reduced mass" $\mu = mM/(m + M)$. (For $M/m \gg 1$ this reduces to m so that center of gravity and laboratory systems are approximately the same.) Suppose the first excitation level of the target nucleus is at an energy W above the ground state. Then if the neutron has a kinetic energy $\mu v_1^2/2 < W$ (center of gravity system) no inelastic scattering can take place. But if $\mu v_1^2/2 > W$ and the nucleus is excited to this level, the kinetic energy of the outgoing neutron after the collision will be

$$(\mu v_0^2/2) = (\mu v_1^2/2) - W$$

so that the cross section becomes

$$\sigma = M' \frac{(\mu v_0)^2}{v_0} \frac{1}{v_1} = M' \mu^2 (v_2/v_1) = M' \mu^2 \sqrt{1 - \frac{2W}{\mu v_1^2}}$$

Writing W as $\mu v_0^2/2$ with v_0 the threshold velocity for excitation of the nucleus to the energy W makes it easier to see how the cross section might be expected to behave near the threshold.

$$\begin{aligned} \sigma &= M' \mu^2 \sqrt{1 - (v_0/v_1)^2} \\ &= M' \mu^2 \sqrt{(v_1 + v_0)/v_1} \sqrt{(v_1 - v_0)/v_1} \end{aligned}$$

For $(v_1 - v_0)/v_1 \ll 1$ (near the threshold) the cross section is approximately $M' \mu^2 \sqrt{(2/v_1)(v_1 - v_0)}$ showing that near the threshold the cross section increases from zero as the square root of the excess velocity, provided M' can be considered constant. Since

the cross section is proportional to v_e , the velocity of the neutron after inelastic scattering it follows that this velocity should likewise increase in proportion to the square root of the excess velocity near the threshold. This is illustrated in figure 7.

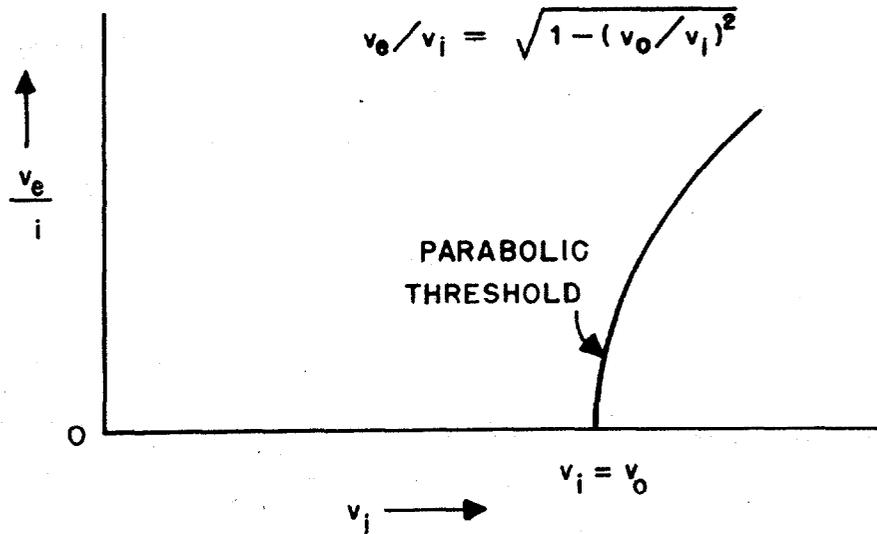


Figure 7. Inelastic scattering of neutrons (v_i = incident neutron velocity, v_e = scattered neutron velocity).

Radiative Capture (n, γ)

The ejected particle is a photon so that instead of p^2/v the quantity h^2v^3/c^3 must be considered, see equation (2-10). If the neutron hits a nucleus and is captured there is generally an excess of energy - that energy associated with the binding of the neutron to the nucleus. Neutron capture is almost always exothermic, one notable exception being helium. The energy available for photon emission will be $\mu v_1^2/2 + W$ where W is the binding energy of the neutron. If the first photon corresponds to a drop to a level L (not necessarily ground) then it will have an energy $h\nu = \mu v_1^2/2 + W_L$. Now as v_1 approaches zero the quantity $h\nu$ approaches a fixed number W_L so that h^2v^3/c^3 becomes constant. The other factor in the cross section is $1/v_1$ which increases rapidly. As a consequence (n, γ) processes should have cross sections which increase as $1/v$ for slow neutrons ($v \sim 0$), provided M' itself is constant.

 (n, α) Processes

These reactions, in which the capture of a neutron results in the emission of an alpha particle, can be endothermic or exothermic (Q positive or negative). As in the previous example of an absorption process the energy equation in the center of gravity system is

$$\mu_0 v_0^2/2 = \mu_1 v_1^2/2 + Q$$

with μ_0 the reduced mass of the ejected alpha particle and μ_1 the reduced mass of the incident neutron. If Q is positive then v_0^2 is at least $(2Q/\mu_0)$ so that the cross section which from equation (2-11) is $M' \mu_0^2 v_0^2/v_1$ is going to obey the $1/v_1$ law, at least for small v_1 . Thus slow neutrons in an (n, α) process for which $Q > 0$ should be absorbed according to a $1/v$ law. However, for $Q < 0$ the situation is different. Denoting the threshold energy by $Q = -\mu_1 v_0^2/2$ the energy equation can be reduced to $(\mu_0/\mu_1)v_0^2 = (v_1^2 - v_0^2)$ with v_0 the threshold velocity. As discussed in the inelastic scattering process the velocity of the emitted alphas should increase in proportion to the square root of the excess velocity at the threshold, Figure 7. Actually, see Figure 8, the rise in σ is not parabolic. This is due to the variation of M' in this case. Being a charged particle the alpha has to escape through an electrostatic potential barrier ("Gamow" barrier). This effectively decreases the cross section as shown in the figure.

2.5 EXAMPLES FROM EXPERIMENT

The absorption of neutrons by boron in the reaction $B^{10} (n, \alpha) Li^7$ illustrates the type process discussed at the end of the preceding section. Q is positive, about 3 Mev (although since Li^7 is normally left in an excited state only about 2.5 Mev are available for kinetic energy). For slow neutrons this reaction should go as $1/v$. Experiment confirms this. In unseparated boron (B^{10} and B^{11}) the cross section has been measured over a wide range of energies with the results shown in Figure 9. For room temperature neutrons the total cross section is 737 barns; since this temperature* ($15^\circ C$) corresponds to a neutron velocity of 2200 m/sec the cross section is thus $737 \times 2200/v = 1.62 \times 10^6/v$ in barns (v in meters per second). For pure B^{10} the room temperature cross section is 3525 barns. This high cross section and the ionizing ability of both products (He^4 and Li^7) as well as the fact that functional dependence of σ on neutron energy E is relatively simple ($\sigma = 116/\sqrt{E}$ for σ in barns E in ev) make boron extremely useful in neutron detectors, particularly in the form of the gas BF_3 . If boron trifluoride is prepared with boron enriched in the isotope B^{10} the detector sensitivity for slow neutrons can be increased, as is apparent from the cross section values, by as much as a factor of five.

*Figure 11 shows the relation between neutron velocities, energies, and temperatures.

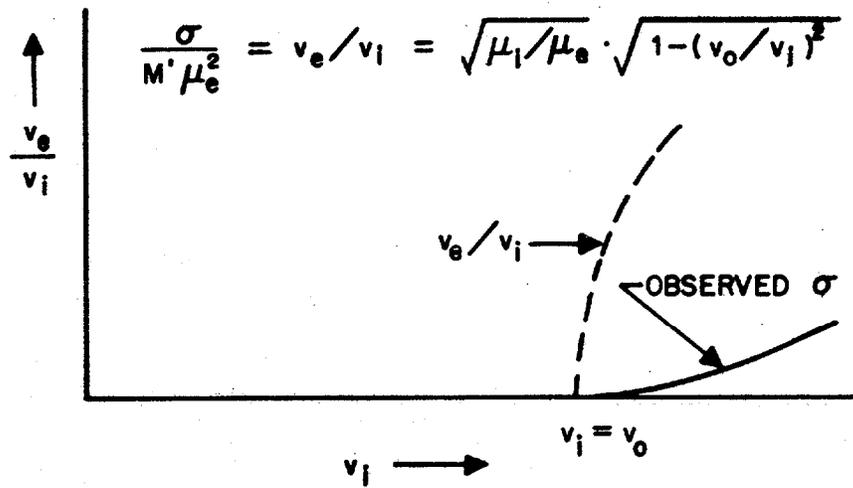


Figure 8. (n, α) process with Q negative. Observed σ not same shape as v_e/v_i at threshold showing M' variation.

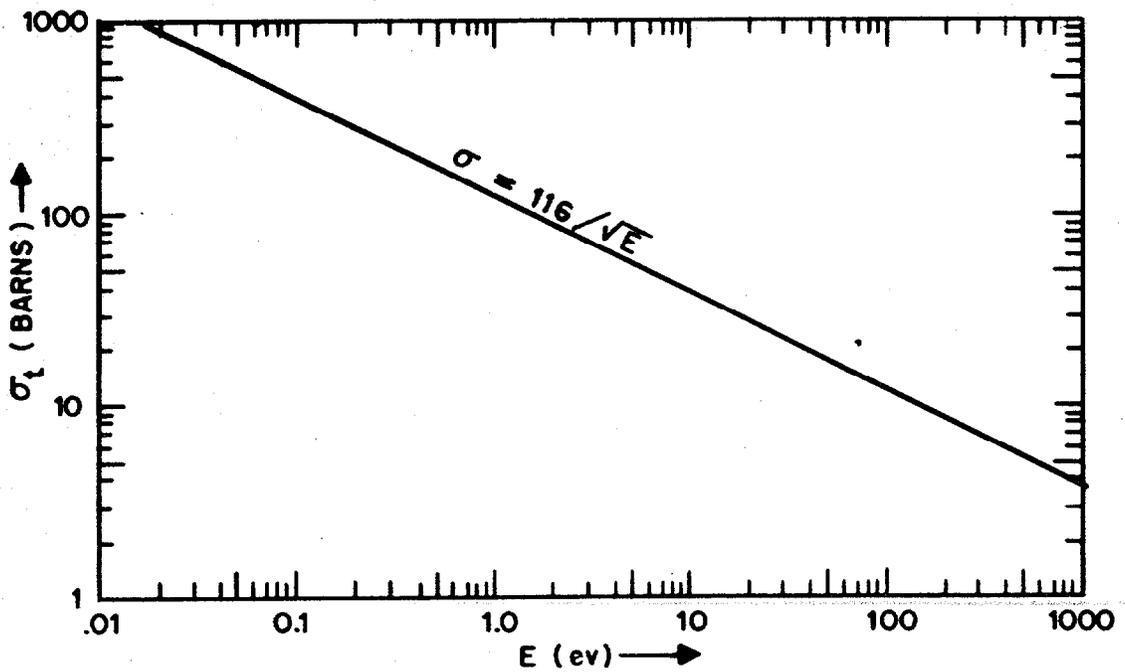


Figure 9. Boron. Total neutron cross section vs. neutron energy.

It is well to keep in mind that the $1/v$ law holds for *relative* velocities. That is, when the neutron velocities become small (comparable to thermal velocities) the thermal agitation of the target nuclei must be considered in the application of the $1/v$ law. Suppose neutrons are incident upon some material in which σ is proportional to $1/v_{rel}$, where the subscripts have been added to indicate that it is the relative velocity of neutron to target nucleus which counts. Suppose further that a fraction N_u of all the target atoms are moving with an absolute velocity u relative to some fixed laboratory frame of reference. Since σ is proportional to $1/v_{rel}$ and the number of "meetings" per second is proportional to $N_u \cdot v_{rel}$, then the capture probability is $A \cdot N_u$ with A a constant. Summing over all possible target atom velocities $\sum_u A \cdot N_u = A \cdot N$ shows that the total capture probability is a constant.

Thus, the number of captures per unit time is a constant and independent of the relative velocities between neutron and target atoms whenever the cross section is proportional to $1/v$. This independence of relative velocities can also be seen by inspection of equation (2-4).

One of the early fundamental experiments (*Physical Review* 49:777 (1936)) was based on this principle. In the experiment the transmission of a rotating boron covered disk, on which a beam of neutrons was directed (axis of neutron beam inclined with respect to axis of disk rotation), was measured. The $1/v$ law was verified by observing no change in transmission with variation of rotational speed. The transmission by substances not obeying the $1/v$ law, e.g., cadmium, was found to vary as the rotation was changed.

The lighter isotope of lithium reacts with neutrons according to the scheme $Li^6(n, \alpha)H^3$ with a Q of +4.5 Mev. As anticipated in the previous section, neutron absorption is according to the $1/v$ law, at least up to about 0.1 ev. This is shown in Figure 10. It may be said in general that the $1/v$ law holds to higher energies for light nuclei, where the energy levels are spaced far apart, than for heavy nuclei where the energy levels are close packed and the factor M' varies sharply. This will be discussed in some detail in Chapter IV.

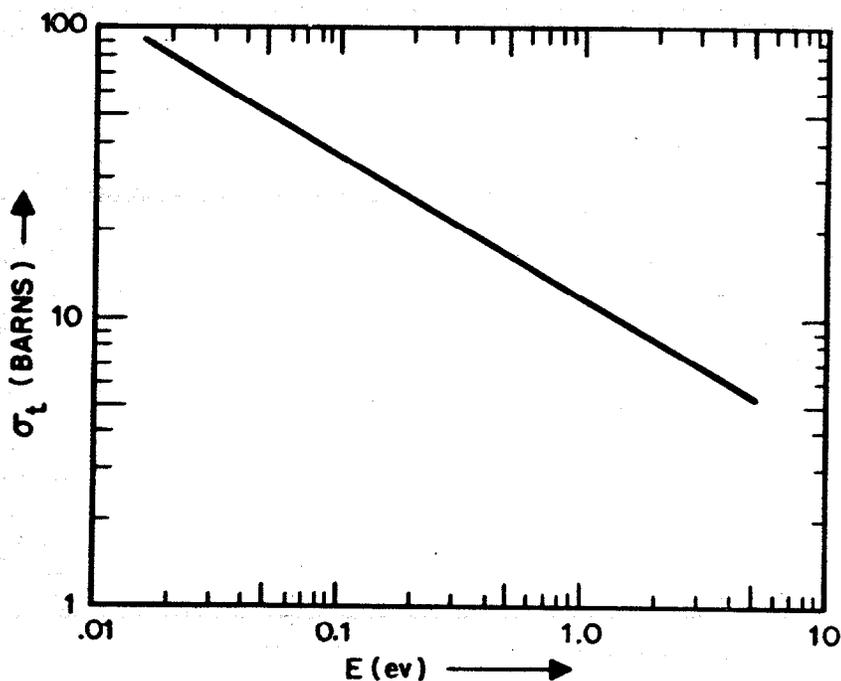


Figure 10. Lithium. Total neutron cross section vs. neutron energy.

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Absorption of neutrons by nitrogen, $N^{14}(n,p)C^{14}$, occurs with the relatively low Q value of 0.6 Mev. This is similar to the case discussed at the end of section 2.5. The $1/v$ law does not hold, being overshadowed by the effect of the Gamow factor. In fact, the cross section is reduced to only a few barns for room temperature neutrons.

Neutron cross sections have been summarized in an article by H. H. Goldsmith, H. W. Ibser, and B. T. Feld in the *Reviews of Modern Physics* 19:259 (1947). They represent part of the increasing body of data of "neutron spectroscopy."

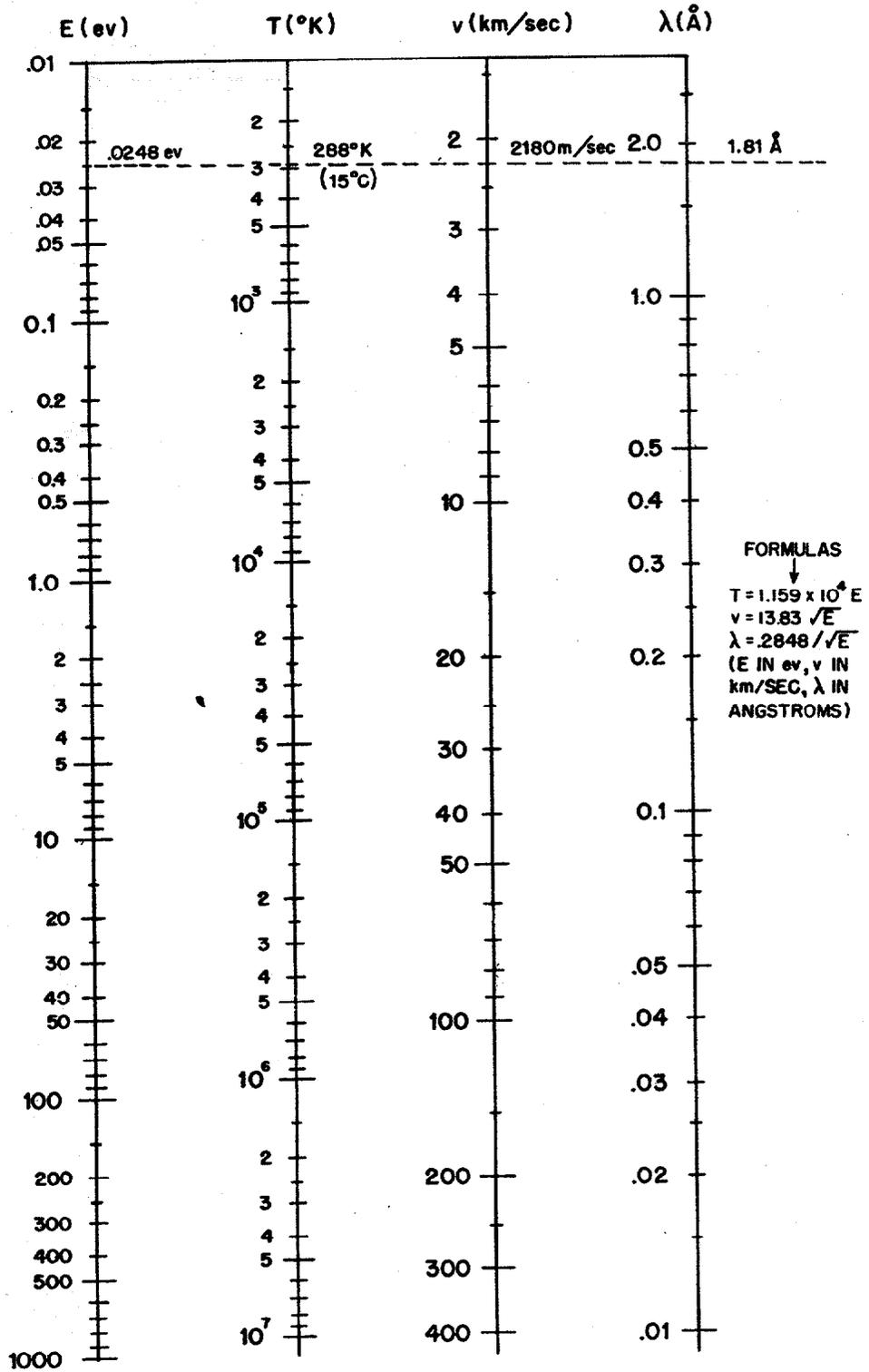


Figure 11.

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PROBLEMS

1. A copper plate 1 cm thick reduces the intensity of a collimated beam of thermal neutrons by a factor 0.36. What is σ_t (for thermal neutrons) in barns for copper? What thickness of copper would reduce the intensity by a factor of 0.5?
2. From the literature find an example of each of the processes: $(n,n)_{\text{elastic}}$, $(n,n)_{\text{inelastic}}$, (n,γ) , (n,α) , (n,p) . Record the cross section observed in each example.
3. What is the reduction in intensity of a beam of one electron-volt neutrons passing through a 50 mg/cm² layer of boron?
4. The "average" distance a neutron goes before being absorbed in a substance whose absorption cross section is σ_a is just $1/N\sigma_a$ (N = target atoms/cc). On this basis what is the average life of a "room temperature" neutron in lithium? (Assume that $\sigma_a = \sigma_{\text{tot}}$) What is the average life in BF₃ at standard conditions?

CHAPTER III

STABLE ISOTOPE CHART AND REACTIONS INVOLVING NEUTRONS

3.1 THE SEGRE ISOTOPE CHART

Before considering in further detail the nature of neutron reactions it will be valuable to review the essential features of stable isotopes and what these features imply about nuclear reactions involving neutrons.

A convenient way to summarize data on the various nuclear species is by means of the Segre Isotope Chart. In this chart the number of neutrons (N) is plotted on the vertical axis, the number of protons (Z) on the horizontal axis. Identity and properties of stable and unstable isotopes are labeled in each square corresponding to the observed (Z, N) values. Since the resulting chart would have the general shape shown in Figure 12 and thereby be inconveniently large (most of the chart would be blank space), the chart is usually broken into sections and the sections arranged in a somewhat interlocking pattern. Isotopes (constant Z) appear in the same vertical column, isobars (constant $Z + N$) on the same diagonal, and isotones (constant N) in the same horizontal row.

The stable isotopes lie in a narrow region on the $(Z + N)$ graph as shown in Figure 12. For light elements this region is centered around the $N = Z$ line; for heavier elements the region deviates toward higher numbers of neutrons so that for uranium the most stable isotope has $N - Z = 146 - 92 = 54$.

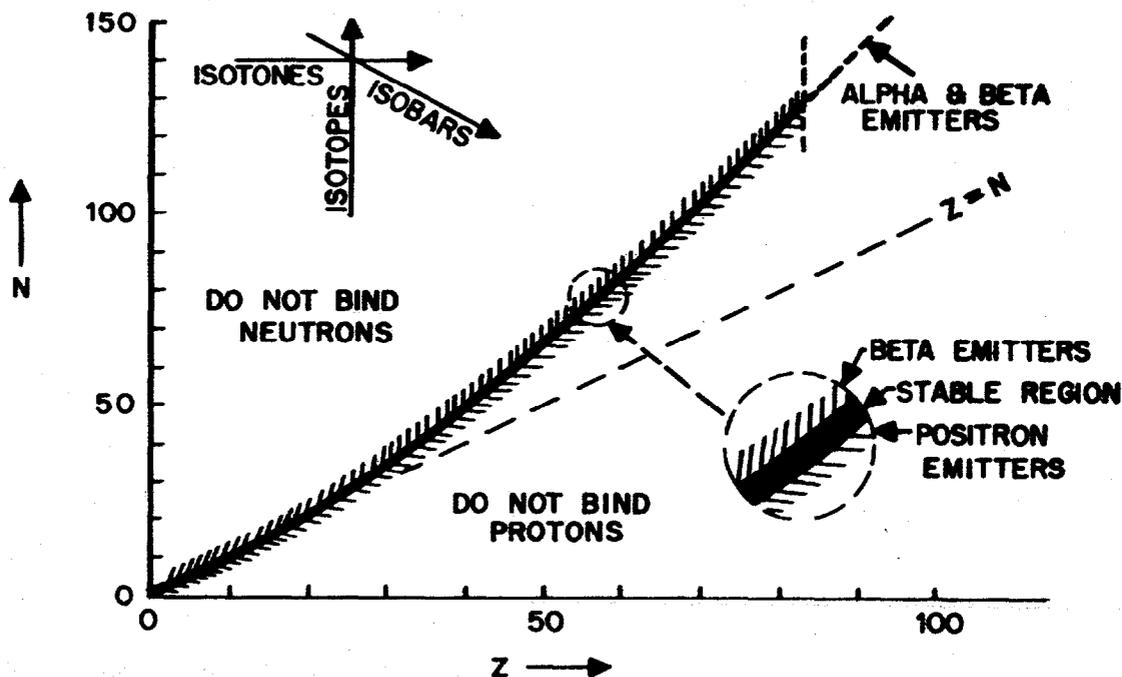
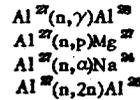


Figure 12. Region of Stable Isotopes

a-93-2'

Z →	Na 11	Mg 12	Al 13	Si 14	P 15
16			$A\beta^-$ 6.7M 28.9895 β^- 2.5	Si 30 3.05 29.9832 $\sigma=0.116$	P 31 100 30.9843 $\sigma=0.23$ $\frac{1}{2}$
15		$\blacktriangle A\beta\gamma$ 10.2M 26.9928 β^- 1.8 γ 1.0	$\bullet A\beta\gamma$ 2.4M 27.9903 β^- 2.75 γ 1.82	Si 29 4.68 28.9866	$A\beta^+$ 2.55M 29.9873 β^+ 3
14	$E\beta\gamma$ 58 s β^- 3.7 γ 0.035	Mg 26 11.41 25.9898 $\sigma=0.048$	Al 27 100 26.9899 $\sigma=0.21$ $+3.634 \frac{5}{2}$	Si 28 92.27 27.9866	$A\beta^+$ 4.6s 28.9919 β^+ 3.63
13	$\blacksquare A\beta\gamma$ 14.84 23.9975 β^- 1.39 σ 138,276	Mg 25 10.18 24.9938	$\blacktriangledown A\beta^+$ 7.0 s 25.9929 β^+ 2.99	$A\beta^+$ 4.5s 26.9949 β^+ 3.64	
12	Na 23 100 22.99618 $\sigma=0.63$ $+2.217 \frac{3}{2}$	Mg 24 78.41 23.9924			
↑ N					

Typical neutron reactions on Al^{27} :



KEY TO SEGRE CHART (AECD 2111)

Element mass number
 Per cent abundance
 Mass
 Slow neutron capture
 Magnetic moment, spin

Class, type radiation
 Half-life
 Mass
 Emitted radiations,
 energy in Mev

STABLE

RADIOACTIVE

Classification:

- A Isotope certain (A and Z certain)
- B Isotope probable, element certain
- C One of few isotopes, element certain
- D Element certain
- E Element probable
- F Insufficient evidence

Type Radiation:

- β^- Negative beta particle
- β^+ Positive beta particle
- γ Gamma ray
- e^- Internal conversion electron
- K Electron capture
- IT Isomeric transition
- σ Slow neutron capture cross section in barns

Figure 13. Segre chart near Al^{27} .

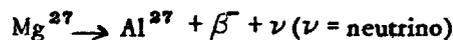
To illustrate the convenience of this representation and to show what data concerning neutron reactions can be obtained from the chart the part of the diagram near aluminum is reproduced in Figure 13. In the figure the single isotope of aluminum, Al^{27} , is outlined with heavy lines and the products of (n, γ), (n,p), (n, α) and (n,2n) reactions on Al^{27} indicated with auxiliary symbols. It turns out that all these reactions lead to unstable end products. Except for the product of the (n,2n) reaction these are all beta emitters. This is understandable, for example, in the case of the (n,p) reaction where when stability is finally achieved the net effect is the transformation of a neutron into a proton and an electron:



The net effect can be written as $n \rightarrow p + \beta^-$. Conversely, there exist (p,n) reactions for which the net result is $p \rightarrow n + \beta^+$. As the neutron mass is greater than that of proton plus electron the transformation of a neutron is exothermic. The converse reaction is endothermic.

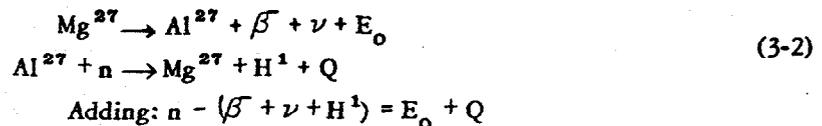
The (n,2n) reaction, not generally as common as the other three, amounts to the extraction of a neutron thereby producing a positron emitter and ultimately stable Mg^{26} . This latter illustrates the principle that unstable nuclei above the curve of maximum stability are generally beta emitters, those below positron emitters (or K-electron "capturers"). Any one of these processes - beta or positron emission or K-capture - produces an isobar of the unstable nucleus so that the processes are along isobaric lines of slope = -1 on the Segre Chart as indicated in Figure 12.

If aluminum is bombarded by neutrons of assorted energies, all the products may appear and some sort of chemical separation would be used to separate the activities. If the energy of the bombarding neutrons is controllable, it might be possible to favor the formation of one product over the others by using neutrons of appropriate energy. In any case isolation of any one of the beta-emitting products and examination of the energy spectrum of the emitted beta particles would reveal that the spectrum is continuous with the general shape shown in Figure 14. This is a somewhat unexpected result inasmuch as gamma emission (and alpha emission of the naturally radioactive nuclei) yields discrete spectra. The accepted explanation of this anomaly is that the emission of a beta particle is always accompanied by the emission of a "neutrino." Thus the decay of Mg^{27} can be written as:



The energy balance is $E_\beta + E_\nu = E_0$, with E_0 the energy of β^- decay and equal to the maximum of the observed beta energy. For Mg^{27} this is 1.8 Mev. Since the energy E_0 can be divided between the beta and the neutrino, it follows that there should be a continuous energy spectrum for the beta particles.

Before such a scheme is acceptable it is necessary to show that the maximum beta energy (E_0), and not, say, the average beta energy, is the energy lost per nucleus on beta decay. This can be readily shown in many instances. For example, consider the decay of Mg^{27} and the production of Mg^{27} :



Now the energy equivalence of the mass difference between neutron and hydrogen atom (proton plus electron) assuming neutrino mass to be negligible is 0.75 Mev so that $E_0 + Q$ should be

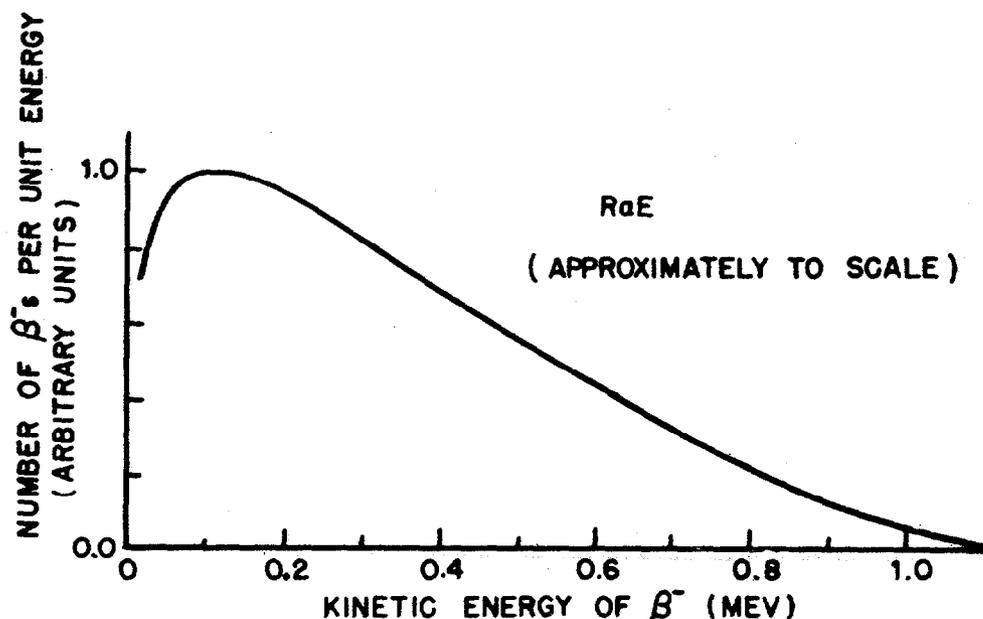


Figure 14. Typical Beta Spectrum.

0.75 Mev. Since E_0 is 1.8 Mev it follows that Q must be -1.05 for consistency. This is observed experimentally, for the reaction $Al^{27}(n,p)Mg^{27}$ is found to be endothermic with a threshold corresponding to $Q = -1.05$ Mev. If the observed Q were, say, -1.2 Mev this would imply that the energy given out upon beta-decay exceeds the maximum of the beta spectrum. In this case to satisfy energy conservation the neutrino mass would have to be assumed non-negligible.

Positron emission is similar in all respects. As with beta emitters the total energy of the reaction must be taken equal to the maximum energy of the emitted particle. Observed spectra and reaction energies are quantitatively consistent provided a neutral particle of negligible mass (i.e., mass small compared to electronic mass) is assumed to accompany the emission of a positron.

3.2 ISOTOPIC WEIGHTS AND THE BINDING ENERGY OF NEUTRONS

In an (n,γ) reaction a neutron is captured by a nucleus and the excess energy emitted as gamma radiation. The energy balance of the reaction leads to a quantitative measure of the binding energy of the neutron to the target nucleus. A rough idea of the magnitude of this energy may be obtained by assuming that the addition of a neutron to a nucleus of mass number A increases its mass number to $A + 1$ and its mass by one unit. Since the neutron mass is, in round numbers, 1.009 mass units this would indicate a binding energy of about .009 mass units or 8 Mev. Actually the true atomic weight differs appreciably from the mass number A in many cases. It has, in fact, been found convenient to define the fractional deviation as the "packing fraction":

$$f(A) = (M-A)/A \quad (3-3)$$

M = atomic weight (a function of A)
 A = mass number

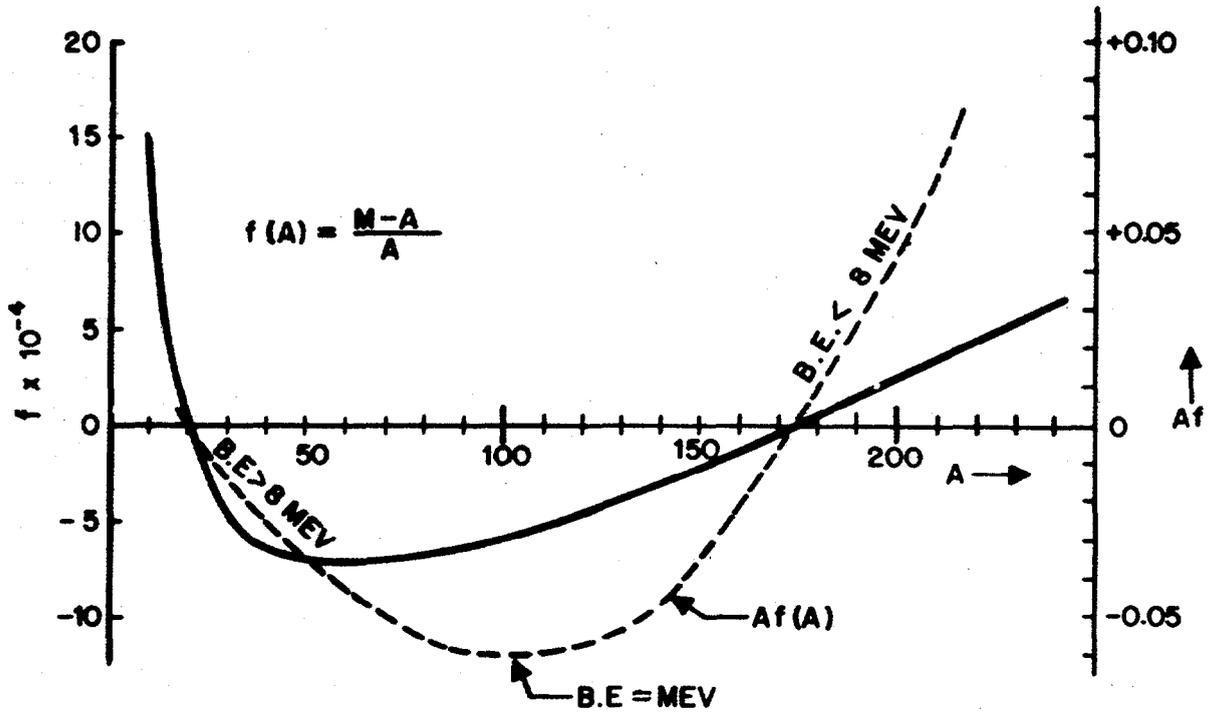


Figure 15. Packing Fraction.

The packing fraction can be determined from mass spectrometric data, solid curve of Figure 15. It can be seen that the packing fraction is slowly varying over all the stable nuclei except the very light ones.

It is possible to use the observed packing fraction curve to determine any regular variation of the binding energy with atomic number. Solving equation (3-3) for $M(A)$ and writing the corresponding equation for $M(A + 1)$, the atomic weight when A is increased by one yields:

$$\begin{aligned}
 M(A) &= A [1 + f(A)] \\
 M(A + 1) &= (A + 1) [1 + f(A + 1)] \\
 \Delta M &= M(A + 1) - M(A) = 1 + (A + 1) f(A + 1) - Af(A)
 \end{aligned}
 \tag{3-4}$$

The ΔM is the increase in atomic mass if one neutron (or proton) is added. Subtracting this from the average mass of neutron and proton, 1.0085, gives the binding energy of the added particle:

$$\begin{aligned}
 1.0085 - \Delta M &= .0085 - [(A + 1) f(A + 1) - Af(A)] \\
 &= .0085 - \frac{d}{dA} [Af(A)]
 \end{aligned}$$

Writing the difference as a derivative makes it possible to estimate the binding energy by observing the slope of the function $Af(A)$. The broken line in Figure 15 shows how $Af(A)$ varies with A . Where $Af(A)$ has a zero slope the binding energy of a neutron (or proton) is .0085 mass units or 8 Mev; where the slope is negative the binding energy is greater than 8 Mev, where positive it is less than 8 Mev. The minimum occurs near $A = 100$.

What this means in so far as neutron reactions are concerned is that (n, γ) reactions will involve energies of the order of 8 Mev. The only stable isotopes which will not bind a neutron are He^4 and the neutron itself. Of course, any unstable isotope with excess neutrons will not bind a neutron. In fact, referring to Figure 12, the region above the beta emitters might be described as "these nuclei will not bind neutrons" and that region below positron emitters as "these nuclei will not bind protons." Examples of nuclei in the upper region, that is, neutron emitters, are found among the nuclei formed as the result of fission.

It should be pointed out that this 8 Mev rule for (n, γ) reactions is just a rough average and that, particularly for light elements, the value may differ from this average. For example, the opposite process or (γ, n) reaction occurs in the case of deuterium (see Chapter I) at a threshold of 2.2 Mev and in the case of beryllium at 1.7 Mev, showing that the neutron binding energies can differ considerably from 8 Mev.

PROBLEMS

1. Find five examples of each of (n, γ) , (n, p) , (n, α) and $(n, 2n)$ processes. Record the half-lives of any radioactive product nuclei.
2. Calculate the binding energy of a neutron to each of the following nuclei: H^1 , H^2 , Li^6 , Li^7 , Be^9 , B^{10} , B^{11} , C^{12} , C^{13} .
3. Calculate the packing fraction and average binding energy of a neutron (or proton) in the neighborhood of iron.
4. Write the equation for the transformation of a proton into a neutron (plus other particles). What is the Q value?

CHAPTER IV

MODELS OF NUCLEI AND OF NUCLEAR REACTIONS

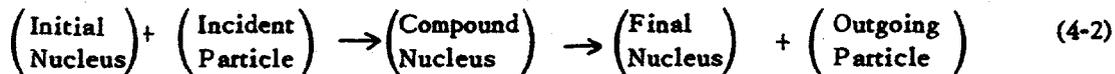
4.1 THE COMPOUND NUCLEUS

In section 2.4 it was shown that for (n, γ) processes the cross section for low energies is:

$$\sigma = M' (h^2 \nu^2 / c^3) \frac{1}{v_i} \quad (4-1)$$

where ν = frequency of emitted gamma, v_i = velocity of incident neutron. Of the more than one hundred (n, γ) reactions known the $1/v$ law holds for most only very close to $v_i = 0$. This is due to the sharp variations of M' associated with resonance phenomena.

An explanation of the nature of these resonances can be made in terms of the so-called "compound nucleus" model of reactions first proposed by Niels Bohr in Nature: 137: 344(1936). In this model a nuclear reaction is a three-step scheme:



This means that when a neutron hits a nucleus it does not knock out the first particle it hits. Instead it distributes its energy among the various members of the nucleus and for some time exists in combination with the original nucleus in a system called the compound nucleus.* Whether a particle is emitted by the compound nucleus depends upon the probability of concentrating the necessary escape energy on one particle in the course of the many-body interactions within the compound nucleus. An (n, γ) process may be represented as:



The compound nucleus $(A + n)^*$ is in this instance the excited state of the final nucleus, the asterisk being used to denote excitation.

The compound nucleus is relatively stable, that is to say the compound nucleus exists for a time long compared to the time it would take a nuclear particle to cross the nucleus. This latter time is of the order of 10^{-12} cm divided by the neutron velocity (see Figure 11, page 21), or for slow neutrons about $10^{-12}/10^7 = 10^{-19}$ second. The time of existence can be inferred from the uncertainty principle which states that the product of the energy uncertainty in a system (ΔE) and the time uncertainty (Δt), in this case the time during which the system can be said to exist, is of the order of Planck's constant:

$$\Delta E \cdot \Delta t \sim h \quad (4-4)$$

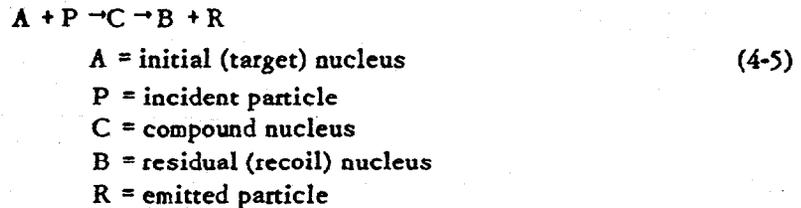
It is possible to determine the magnitude of the energy uncertainty to be discussed later, so that Δt can be inferred. Sometimes Δt is of the order of 10^{-14} second or 10^5 times the crossing time. In terms of the physical picture this means that the incident particle hits the nucleus, distributes its energy among the nuclear particles, and very many "transit times" later the necessary energy is concentrated in a constituent particle so that it can escape.

* The distribution of energy among all nuclear particles arises from the fact that the nuclear particles interact with forces comparable to the force exerted upon any nuclear particle by the incident neutron. This is different from the case in atomic collisions when, say, an electron hits an atom. In this latter case the interaction between the colliding electron and the electron being struck is large compared to the interaction between the struck electron and the other electrons of the atom. Put another way, collisions between neutrons and nuclei must be considered "many-body" problems whereas electron-atom collisions can generally be reduced to a two-body (or one body in a field) problem. For an excellent discussion of these differences and a general introduction to nuclear processes see H. A. Bethe in Reviews of Modern Physics 9:71 to 74 (1937).

4.2 NEUTRON RESONANCES, LEVEL SPACING

The quantity M' in equation (4-1) depends upon certain matrix elements H as described in section 2.4. These matrix elements in turn depend upon the possible states of the intermediate or compound nucleus in such a way that when the sum of the incident particle's kinetic energy and binding energy is equal (or nearly equal) to the energy of some excited state of the compound nucleus then the factor M' becomes large resulting in a peak in the absorption cross section for this particular kinetic energy of the incident particle. When the particle's kinetic energy is such that the total energy available to the compound nucleus is different from any energy corresponding to an excited state of the compound nucleus then the factor M' is relatively constant for variation in incident particle energy.

Putting these observations symbolically the general nuclear process can be written:



with the energy relations:

$$\text{(Conservation of Energy)} \quad W_A + W_P + E_P = W_B + W_R + E_R$$

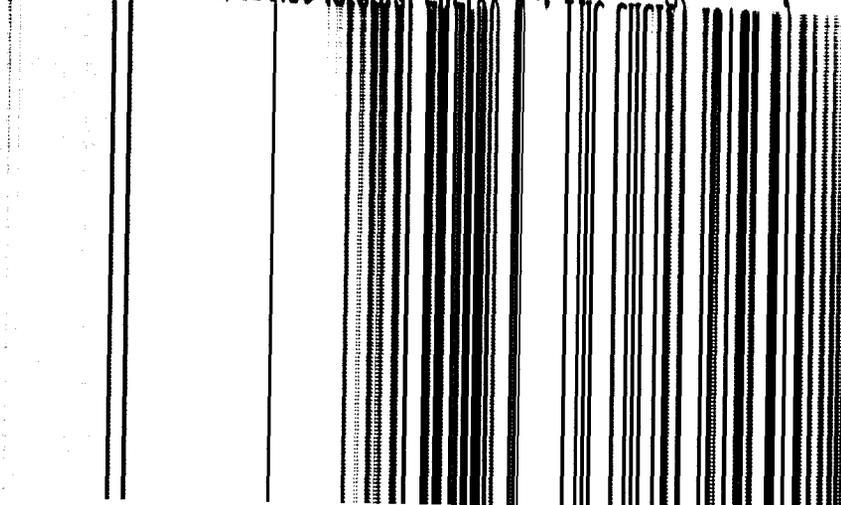
$$\text{(Definition of } E_{Pr}) \quad W_A + W_P + E_{Pr} = W_{Cr}$$

(4-6)

- W_A = internal energy of initial nucleus
 W_P = internal (binding) energy of incident particle
 E_P = kinetic energy of incident particle
 W_B = internal energy of residual nucleus
 W_R = internal (binding) energy of emitted particle
 E_R = kinetic energy of emitted particle
 E_{Pr} = kinetic energy of incident particle when that kinetic energy is just equal to that necessary to bring the compound nucleus to an excited state characterized by internal energy W_{Cr}
 W_{Cr} = internal energy of compound nucleus C at resonance r.

Perhaps the best way to appreciate the significance of equations (4-5) and (4-6) is to consider the (n, γ) process. A is the target nucleus, P the neutron, B the residual nucleus (isotopic to A but with one unit increase in mass number), and R the emitted gamma photon. Assume A is in the

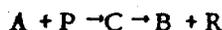
..... depending internal energy W_A . The energy level system of the initial state,



4.2 NEUTRON RESONANCES, LEVEL SPACING

The quantity M' in equation (4-1) depends upon certain matrix elements H as described in section 2.4. These matrix elements in turn depend upon the possible states of the intermediate or compound nucleus in such a way that when the sum of the incident particle's kinetic energy and binding energy is equal (or nearly equal) to the energy of some excited state of the compound nucleus then the factor M' becomes large resulting in a peak in the absorption cross section for this particular kinetic energy of the incident particle. When the particle's kinetic energy is such that the total energy available to the compound nucleus is different from any energy corresponding to an excited state of the compound nucleus then the factor M' is relatively constant for variation in incident particle energy.

Putting these observations symbolically the general nuclear process can be written:



- (4-5)
- A = initial (target) nucleus
 - P = incident particle
 - C = compound nucleus
 - B = residual (recoil) nucleus
 - R = emitted particle

with the energy relations:

$$\text{(Conservation of Energy)} \quad W_A + W_P + E_P = W_B + W_R + E_R$$

$$\text{(Definition of } E_{Pr}) \quad W_A + W_P + E_{Pr} = W_{Cr} \tag{4-6}$$

- W_A = internal energy of initial nucleus
- W_P = internal (binding) energy of incident particle
- E_P = kinetic energy of incident particle
- W_B = internal energy of residual nucleus
- W_R = internal (binding) energy of emitted particle
- E_R = kinetic energy of emitted particle
- E_{Pr} = kinetic energy of incident particle when that kinetic energy is just equal to that necessary to bring the compound nucleus to an excited state characterized by internal energy W_{Cr}
- W_{Cr} = internal energy of compound nucleus C at resonance r.

Perhaps the best way to appreciate the significance of equations (4-5) and (4-6) is to consider the (n, γ) process. A is the target nucleus, P the neutron, B the residual nucleus (isotopic to A but with one unit increase in mass number), and R the emitted gamma photon. Assume A is in the ground state with corresponding internal energy W_A . The energy level system of the initial state, see Figure 16, will be a continuum since the incident neutron kinetic energy (E_P) can vary continuously. When the energy of the neutron approaches certain critical values E_{P2}, E_{P3}, \dots the total energy of the initial state corresponds to the energy levels of the compound nucleus W_{C2}, W_{C3}, \dots , respectively, and the probability of neutron absorption increases as shown in the lower graph of Figure 16. Now since the compound nucleus when excited to any one of these levels can get rid of its excess energy by more than one process, e.g., $(n, \gamma), (n, \alpha), (n, p)$, it follows that all the resonance peaks observed in the cross section do not necessarily correspond to an (n, γ) process. For medium weight nuclei one can be certain from energy considerations and the effect of the Coulomb potential barrier that at least the first few levels do correspond to true neutron capture. Of course, neutrons with large kinetic energies may bring the compound nucleus to excited states where several processes compete in achieving ultimate stability.

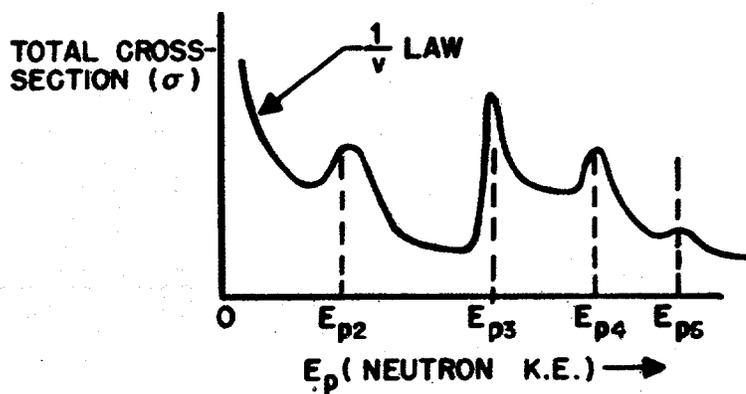
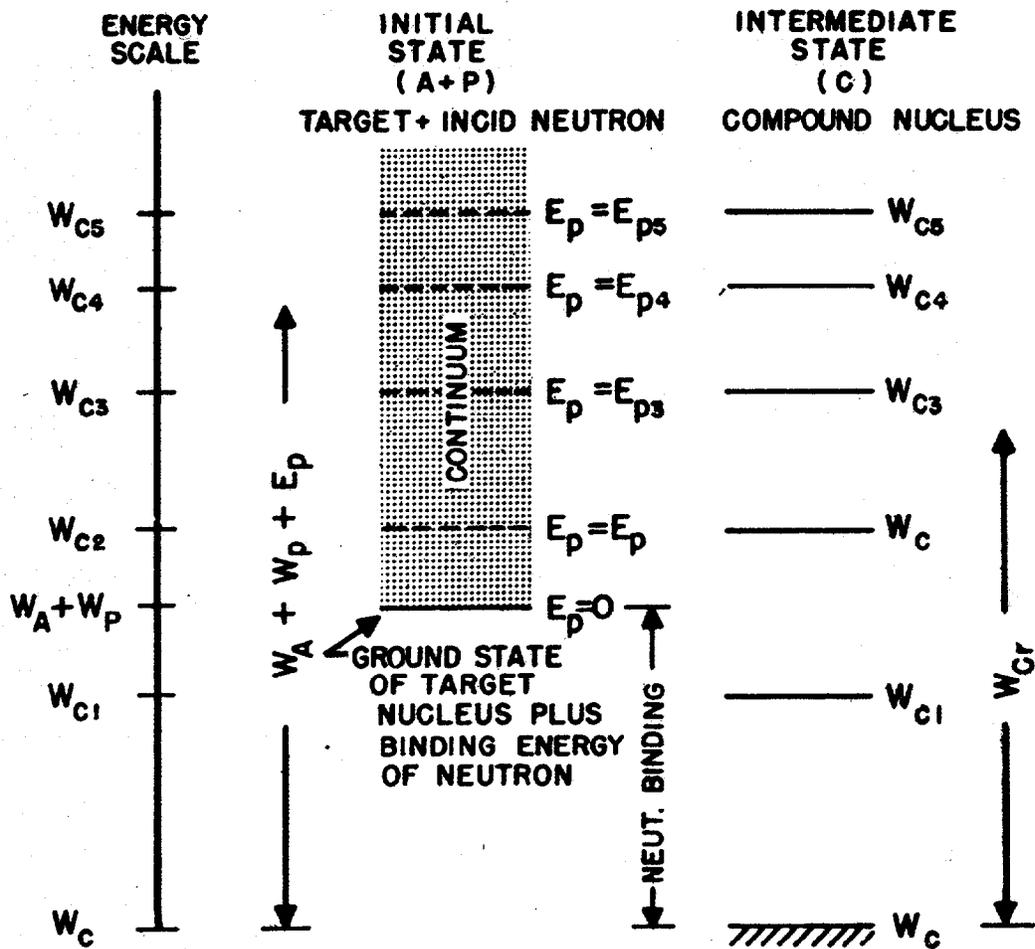


Figure 16. Level System in Compound Nucleus

If the first resonance peak occurs on the average for, say, nuclei of medium weight at $E_p \sim E$, then it would be anticipated that the energy level spacing of medium weight nuclei (in the region of excitation corresponding to the binding energy of the incident particle) is of the order of magnitude of E . For medium weight nuclei such "first capture resonances" occur at ~ 10 electron volts, so that the energy level spacing at compound nucleus energies ($W_A + W_p \sim 8$ Mev) is of the order of 10 electron volts. Experiments with such nuclei that are near their ground states (e.g., excitation by gammas) indicate that the spacing between energy levels is of order of 10^5 or 10^6 electron volts at low energies.

4.3 TWO NUCLEAR MODELS

The fact that the level spacing decreases very rapidly with excitation energy for medium (or large) weight nuclei needs explanation. Qualitatively this fact can be appreciated by considering two models of the compound nucleus: (1) a mechanical system with many degrees of freedom (2) a neutron-proton gas.

In the first model the nucleus is represented by a mechanical system with A members ($A =$ mass number), each member having three degrees of freedom. Since the total number of degrees of freedom is $3A$ the system will sustain vibrations with $3A$ characteristic frequencies. With a vibration frequency ν there is associated an energy $h\nu$. If the system is vibrating with frequencies ν_1 and ν_2 at once it has an energy $h\nu_1 + h\nu_2$. In general it will have an energy:

$$a_1 h\nu_1 + a_2 h\nu_2 + a_3 h\nu_3 + \dots + a_{3A} h\nu_{3A}$$

where the a 's are integers. If $\nu_1 = \nu_2 = \nu_3$, etc., it is easy to see that at high energies there will be more levels per unit energy. This is strictly an inexact qualitative argument but the basic idea, that a system of many degrees of freedom will pack its energy levels at high energy, is correct.

This type of consideration can be refined by depicting the nucleus as a liquid drop (Bohr and Kalckar proposal: Kgl. Dansk Acad. Vol. 14, No. 10, (1937)). The liquid drop is held together by the attraction of each part of the drop to the nearest neighboring part of the drop, that is by short range forces (short compared to droplet dimensions). Drop volume is proportional to drop mass. Similarly nuclear forces are short range forces with each proton attracted to the very few neighboring neutrons or protons and the nuclear volume is proportional to the total number of neutrons and protons in the nucleus. Considering a spherical drop there are a number of possible modes of vibration. Higher modes will be characterized by a relatively "wrinkled" surface, that is many nodal lines; the energies corresponding to these higher modes will be relatively closer together than those for lower and fundamental modes. This liquid drop model will be discussed later in the chapter on fission.

The second nuclear model is a Fermi gas of neutrons and protons in a potential well. The temperature of the gas is T . When $T = 0$ the nucleus is in the ground state with some particles moving rather fast nonetheless because of the Pauli exclusion principle. If energy is fed into the gas some particles move faster and T increases. It can be shown that for a degenerate gas of the type here considered the energy is proportional to the T^2 rather than T . In fact,

$$\text{Energy} = U = (\pi^2/4)(A/\zeta)\tau^2$$

$A =$ number of particles

$\tau = kT$

$\zeta =$ energy of ground state

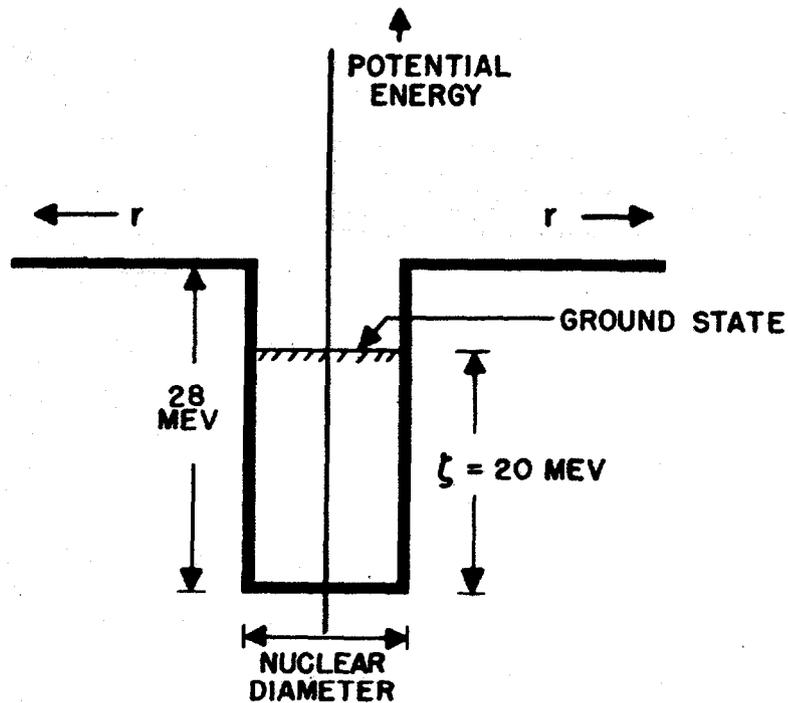


Figure 17. Fermi gas model: potential well

In the potential well chosen for the nuclear model, Figure 17, ζ is ~ 20 Mev. Writing U as aT^2 the entropy $S (= \partial U / \partial \tau)$ is just $2aT$ or $\pi\sqrt{AU/\zeta}$. The entropy is customarily written as $\log P + \text{constant}$ where P is the probability of the state in question or statistically the number of states per unit energy interval. Thus

$$S = \log P + \text{constant}$$

or
$$P = \text{constant} \cdot e^S = P_0 e^{\pi\sqrt{AU/\zeta}}$$

with P_0 the value of P at $U = 0$. Applying this to a medium nucleus with 8 Mev excitation ($A \sim 100$, $\zeta \sim 20$ Mev) shows that the density of states is $P_0 e^{20}$ or $P_0 \cdot 10^8$. That is the density of states (number of states per unit energy) at 8 Mev is 10^8 times what it is at the ground state. This is somewhat too high.

In conclusion it should be pointed out that the foregoing only indicates trends and should not be taken for more than it is worth. The essential idea is that the nucleus is a system with many degrees of freedom. Almost any approach shows logarithmic variation in level density. Achieving numerical consistency is an almost impossible task because of the lack of fundamental knowledge of nuclear forces and the lack of mathematical apparatus for handling what is essentially a many-body problem.

The reader is referred to Bethe's "Nuclear Physics, Part B" (Rev. Mod. Phys. 9(1937)) section 53, pages 79 to 90, for a summary of the various approaches to the problem of nuclear energy level distributions.

PROBLEMS

1. Find all the information you can in the literature about indium and gold resonances for neutrons.
2. From the literature find the first resonances for nuclei with mass numbers $A = 100$ to $A = 150$. Make an estimate of the level spacing in this range.
3. Make a table of level density for Be at 4, 6, 8, 10 Mev excitation using the neutron-proton gas model. Do the same for Fe, Ag, Au.

CHAPTER V

THE SCATTERING OF NEUTRONS

5.1 THE BREIT-WIGNER FORMULA

For certain special cases, the matrix elements occurring in the general formula for the cross section, that is, the factor M discussed briefly in section 2.4, can be reduced to relatively simple formulas. In particular, if the resonance levels of a compound nucleus are far apart and we are interested in the cross section in the neighborhood of one of the resonance levels, the expression for the cross section of an (a,b) process can be shown to take the form:

$$\sigma(a,b) = \pi \kappa_a^2 \frac{\Gamma_a \Gamma_b}{(E_a - E_r)^2 + (\Gamma^2/4)} \quad (5-1)$$

$$\kappa_a = \lambda_a/2 = \pi/\sqrt{2\mu E_a}; \quad \mu = mM/(m+M)$$

E_a is the kinetic energy of the incident particle, λ_a its deBroglie wavelength, m the mass of the incident particle, M the mass of the target nucleus, E_r is the resonance level energy (E_{Pr} in Section 4.2) of the compound nucleus, and Γ the width of the resonance peak at half its maximum value. Actually equation (5-1), known as the Breit-Wigner formula, should be multiplied by factors depending on the spins of the initial particles and the compound nucleus. For simplicity, consider these factors to be incorporated in the Γ_a and Γ_b . Γ_a and Γ_b are the partial widths of the resonance peak and are associated with the probability of emitting "a" and "b" particles, respectively. Their exact form is rather complicated, but since the probability of emission of a particle "b" of momentum p_b [see equation (2-11)] is proportional to p_b^2/v_b , then so is Γ_b . The relative probability that the outgoing particle will be an "a" particle is Γ_a/Γ , relative probability that it will be a "b" particle, Γ_b/Γ , etc., so that the sum of all Γ_i is Γ :

$$\sum \Gamma_i = \Gamma \quad (5-2)$$

$$i = a, b, \dots$$

If τ_i is the average time of emission of a particle i after the formation of the compound nucleus, then

$$\Gamma_i \tau_i \approx \pi \quad (5-3)$$

This is a restatement of the uncertainty relation discussed in section 4.1 and expressed in equation (4-4). The probability that no particle i has been emitted from the nucleus up to a time t is $\prod_{i=a,b,\dots} \exp(-t/\tau_i) = \exp(t \sum 1/\tau) = e^{-t/\tau}$

where t is the average life of the compound nucleus:

$$\tau = [\sum_i 1/\tau_i]^{-1} \quad (5-4)$$

If one describes the emission in terms of the disintegration constant $\lambda = 1/\tau$ the exponential relation is seen to be the familiar $\exp(-\lambda t)$. The average life τ is seen to satisfy the Heisenberg relation in the same manner as the τ_p that is, $\Gamma_\tau \approx \pi$.

The Breit-Wigner formula of equation (5-1) can be applied to any nuclear collision involving the formation of a compound nucleus, provided that the resonance levels are not so close together

that they appreciably distort this one-level formula. It can be applied, for example, to (n, γ) processes. For resonance capture in indium, it is observed* that $E_r = 1.44$ ev, $\sigma(n, \gamma)$ at resonance $\approx 26,000$ barns, $\Gamma \approx 0.09$ ev. In addition, the experimental measurements show that only neutrons and gammas are observed to be emitted (thus $\Gamma = \Gamma_n + \Gamma_\gamma$), and more gammas are emitted than neutrons ($\Gamma_\gamma > \Gamma_n$). With these experimental data, it is desired to find the number of neutrons emitted for each gamma emitted. The solution follows from equation (5-1):

$$\sigma(n, \gamma) \text{ at resonance} = 4\pi \lambda_n^2 \Gamma_n \Gamma_\gamma / \Gamma^2$$

$$\lambda_n = \pi / \sqrt{2mE_r}; \quad m = \text{Neutron mass}$$

Substituting the experimental values of σ and E_r gives the value

$$\Gamma_n \Gamma_\gamma / \Gamma^2 = 0.015$$

Since $\Gamma_n < \Gamma_\gamma$, then $\Gamma_\gamma \approx \Gamma = 0.09$ ev. Also $\Gamma_n \Gamma_\gamma / \Gamma^2 \approx \Gamma_n / \Gamma = 0.015$, from which $\Gamma_n = 0.015 \Gamma = 0.015 \times 0.09 = 0.0013$ ev. Thus the ratio of widths is $\Gamma_n / \Gamma_\gamma = 0.015$, or for each thousand gammas emitted, approximately 15 neutrons will be emitted.

The same reasoning for gold and silver† shows that:

$$\text{Au: } E_r = 4.8 \text{ ev, } \sigma_{\text{res}} \approx 60,000 \text{ barns, } \Gamma \approx 0.1 \text{ ev (experimental)}$$

$$\Gamma_n \approx 0.01 \text{ ev, } \Gamma_\gamma \approx 0.1 \text{ ev, } \Gamma_n / \Gamma \approx 0.11 \text{ (calculated)}$$

$$\text{Ag: } E_r = 5.1 \text{ ev, } \sigma_{\text{res}} = 7200 \text{ barns, } \Gamma = 0.19 \text{ ev (experimental)}$$

$$\Gamma_n = 0.0027 \text{ ev, } \Gamma = 0.19 \text{ ev, } \Gamma_n / \Gamma = 0.014 \text{ (calculated)}$$

The capture reactions for indium, gold, and silver are very useful in methods of slow neutron detection.††

5.2 SOME GENERAL CONSIDERATIONS ON NEUTRON SCATTERING

For the elastic scattering of low-energy neutrons, formula (5-1), by virtue of the relations $a = b = n$ and $E_n \approx 0$, reduces to

$$\sigma(n, n) = \pi \lambda_n^2 \Gamma_n^2 / E_r^2 \quad (5-5)$$

$$[E_n \approx 0, \Gamma \ll E_r]$$

The inequality is assumed and limits the applicability to neutrons whose energy is less than the first resonance energy and to cases where the width of the first resonance energy level is much less than the energy itself. Now since λ_n is proportional to $1/\mu v_n$ and Γ_n is proportional to p_e^2 / v_e or $\mu^2 v_e$ where p_e and v_e are momentum and velocity of the neutron, and since $v_n = v_e$ (elastic collision), then it follows that $\lambda_n^2 \Gamma_n^2$ is proportional to μ^2 and should be fairly constant for scattering of neutrons with energies less than the first resonance energy.

*Physical Review 71:165 (1947).

†Phys. Review 70:166 (1946); 71:165 (1947); 71:757 (1947).

††For a summary of activation cross sections for thermal neutrons, see L. Seren, H. N. Friedlander, S. H. Turkel in Phys. Rev. 72:888 (1947).

The curve for scattering in hydrogen (paraffin) shown in Figure 18 exhibits a fairly constant cross section of 21 barns at low neutron energies. However, at very low neutron velocities, there is a sudden increase of σ_s . This is explicable in terms of the variation of μ^2 . For neutrons scattered from hydrogen, μ is half the neutron mass *except* where the relative neutron-proton velocity is too small to provide sufficient energy to free the hydrogen (bound to the paraffin) or excite the molecule. In this latter instance, which obtains for very slow neutrons, the hydrogen bound in the large molecule has an effective mass that is very large. Thus

$$\mu = \frac{1}{2} \text{ for } E_n > \text{ binding energy of H to paraffin}$$

$$\mu = 1 \text{ for } E_n \ll \text{ binding energy of H to paraffin}$$

So that the Γ_n will stand in the ratio of the squares of μ , or 1:4. This will increase the scattering cross section for very slow neutrons. In Figure 18, it is seen that experiment confirms that this increase is by a factor of 4 so that the neutron scattering cross section (measured in chilled paraffin*) is about 80 barns. This result may be stated as follows. *The scattering cross section for slow neutrons incident on bound protons is four times that for free protons.*

5.3 SCATTERING BY A POTENTIAL

In cases where resonance effects are negligible, the problem of elastic scattering of neutrons can be treated by considering the target nucleus to be replaced by a potential well. The analysis, then carried through by the usual methods of Schroedinger theory, turns out to yield results valid for thermal neutrons scattered by many elements. It is possible to determine accurate information on σ_s even though the shape of the potential is not known with any certainty.

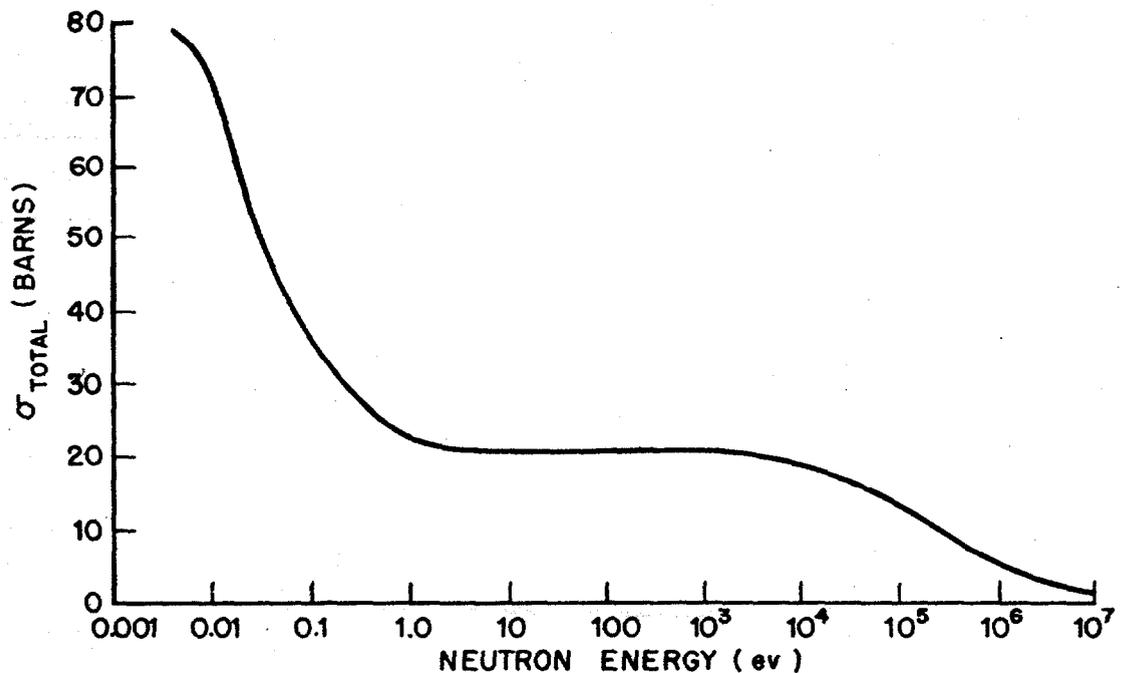


Figure 18. Hydrogen Cross Section [Rev. Modern Phys. 19:260 (1947)].

*Also measured using graphite filter, See Phys. Rev. 70:815 (1946).

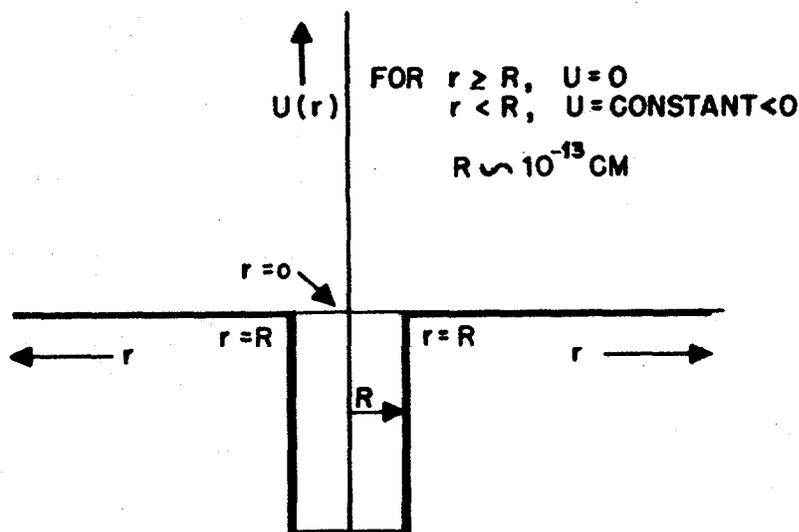


Figure 19. Potential Well Representation of Neutron-Nucleus Interaction.

Figure 19 shows the general shape of the potential well. The wave function ψ of the incident particle satisfies the Schrodinger equation:

$$\nabla^2 \psi + (2m/\hbar^2)[E - U(r)] \psi = 0 \quad (5-6)$$

Where E is the incident particle energy and U the nuclear potential. Considering only s scattering (zero angular momentum), the Schrodinger equation reduces to the radial equation:

$$\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\psi}{dr} \right) + \frac{2m}{\hbar^2} [E - U(r)] \psi = 0 \quad (5-7)$$

or

$$u'' + (2m/\hbar^2)(E - U)u = 0$$

with

$$u = r\psi \text{ and } u'' = d^2u/dr^2$$

The latter simplified form is derived from the relations $u'' = (r\psi)'' = (r\psi' + \psi)' = r\psi'' + 2\psi' = (1/r)(r\psi')'$ where the primes indicate differentiation with respect to r .

Now in the scattering problem at hand, equation (5-7) must be solved for the particular form of $U(r)$ chosen to represent the nucleus with the boundary conditions $\psi=0$ at $r = \infty$ and ψ finite elsewhere. Inspection of (5-7) shows that whenever $E - U > 0$, the curvature of u is negative (u curves toward r axis), that is, $u''/u = - (2m/\hbar^2)(E - U) < 0$. For $E - U < 0$, the curvature $u''/u > 0$, and for $E - U = 0$, it follows that $u''/u = 0$. Referring to Figure 19, it is apparent that there are three cases to consider: (a) $E > 0$, in which case $E - U > 0$ for all r , (b) $E = 0$, for which $E - U > 0$ within the nucleus and $E - U = 0$ outside (c) $E < 0$, in which case $E - U > 0$ inside the nucleus and $E - U < 0$ outside. The solutions for the three classes of values of $(E - U)$ can be readily determined to be:

$$\begin{aligned}
 E - U > 0 \quad u &= A \sin [\sqrt{2m(E-U)/\hbar^2} r] + B \cos [\sqrt{2m(E-U)/\hbar^2} r] \\
 E - U = 0 \quad u &= A' r + B' \\
 E - U < 0 \quad u &= A'' \exp [-\sqrt{2m(U-E)/\hbar^2} r] + B'' \exp [+ \sqrt{2m(U-E)/\hbar^2} r]
 \end{aligned}
 \tag{5-8}$$

Boundary conditions determine the values of A, B, A', B', etc. These conditions are that u/r ($=\psi$) is finite everywhere and vanishes at infinity. In addition, the first derivative must be continuous.

With these conditions and the solutions of (5-8) in mind, Figure 20 can be constructed. In all instances, u must be zero at $r = 0$, since $u/r = \psi$ must remain finite. The variation of u with r is fairly straightforward for the cases $E > 0$ and $E = 0$. The situation when $E < 0$ needs explanation. Near the origin, the usual oscillation is observed with increasing period as the function $E - U$ decreases. At $r = r_0$ (see Figure 20), the oscillation stops, and at greater values of r , the exponential solution, last equation, (5-8), must reduce to a single *negative* exponential, since the positive exponential would not satisfy the condition that u/r is finite at $r = \infty$. That this reduction to a single negative exponential is not possible for all values of E is shown in Figure 20 ($E < 0$) where for $E = E_1$, the coefficient of the positive exponential is negative and for $E = E_2$, the coefficient is positive. Between E_1 and E_2 , there must be some value of E for which the coefficient vanishes. There may be a number of values of E for which u/r is finite at $r = \infty$. These are the allowed values of E for $E < 0$ corresponding to the discrete spectrum or the bound states of the system.

The case $E > 0$ corresponds to the case of an incident particle (positive kinetic energy). As shown in Figure 20, the wave function outside the nuclear radius is a sine function $A \sin(\sqrt{2mE/\hbar^2} r + \delta)$ where δ is a phase shift dependent on the wave function within the nucleus to which the sine must be joined (at $r = R$). The sine function does not, when extrapolated, seem to come from the origin (dotted line figure) but appears to have its origin at a distance "a" from $r = 0$. This distance is related to δ by the equation $a/\lambda = \delta/2\pi$ with $\lambda = 2\pi\hbar/\sqrt{2mE}$ (the de Broglie wavelength of the incident particle).

Now it can be proved* that the scattering cross section is directly dependent on this phase shift δ in such a way that when δ is small (or an integral multiple of π), the scattering cross section is small, and when δ is $\pi/2$ (or an integral multiple of $\pi/2$), the cross section is a maximum. The relation between δ and σ_s is:

$$\sigma_s = (4\pi\hbar^2/m^2v^2) \sin^2 \delta \text{ (s scattering only)} \tag{5-9}$$

The limitation to s scattering means that the incident particle has zero angular momentum. On a classical basis, a particle with velocity v at large distances from the nucleus moving in such a direction that it would pass the nucleus (if unaffected by nuclear forces) at a distance b , Figure 21, has an angular momentum mvb . According to quantum mechanical principles, this must be quantized, or $mvb = l\hbar$ ($l = 0, 1, 2, \dots$). Thus $b = \hbar/mv$, or $b = l\lambda$ ($\lambda =$ de Broglie wavelength $\times 2\pi$). The region between $l = 0$ and $l = 1$ or $b = 0$ to be $= \lambda$ is the region of s scattering. Between $b = \lambda$ and 2λ is the p scattering region. Now if the nuclear size is less than λ , that is, $R < \lambda = \hbar/mv$, then it is obvious that no p scattering is possible. Particles passing at "p distances" from the nucleus will not be aware of the nucleus, due to the short range character of nuclear forces. Recalling the discussion in section 2.2, it is apparent then that *there is s scattering only if the neutrons are slow.*

For very low velocities, the formula (5-9) can be simplified. This is due to the fact that the wave function inside the nucleus will change very little with changes in E when E is small. For this reason, "a" does not vary. However, λ increases as $E^{\frac{1}{2}}$, with the net result that a/λ becomes

*Rasetti, F. "Elements of Nuclear Physics," 1936. Prentice-Hall, p. 204ff; Mott, N.F. and H.S.W. Massey, "Theory of Atomic Collisions," Oxford, 1933.

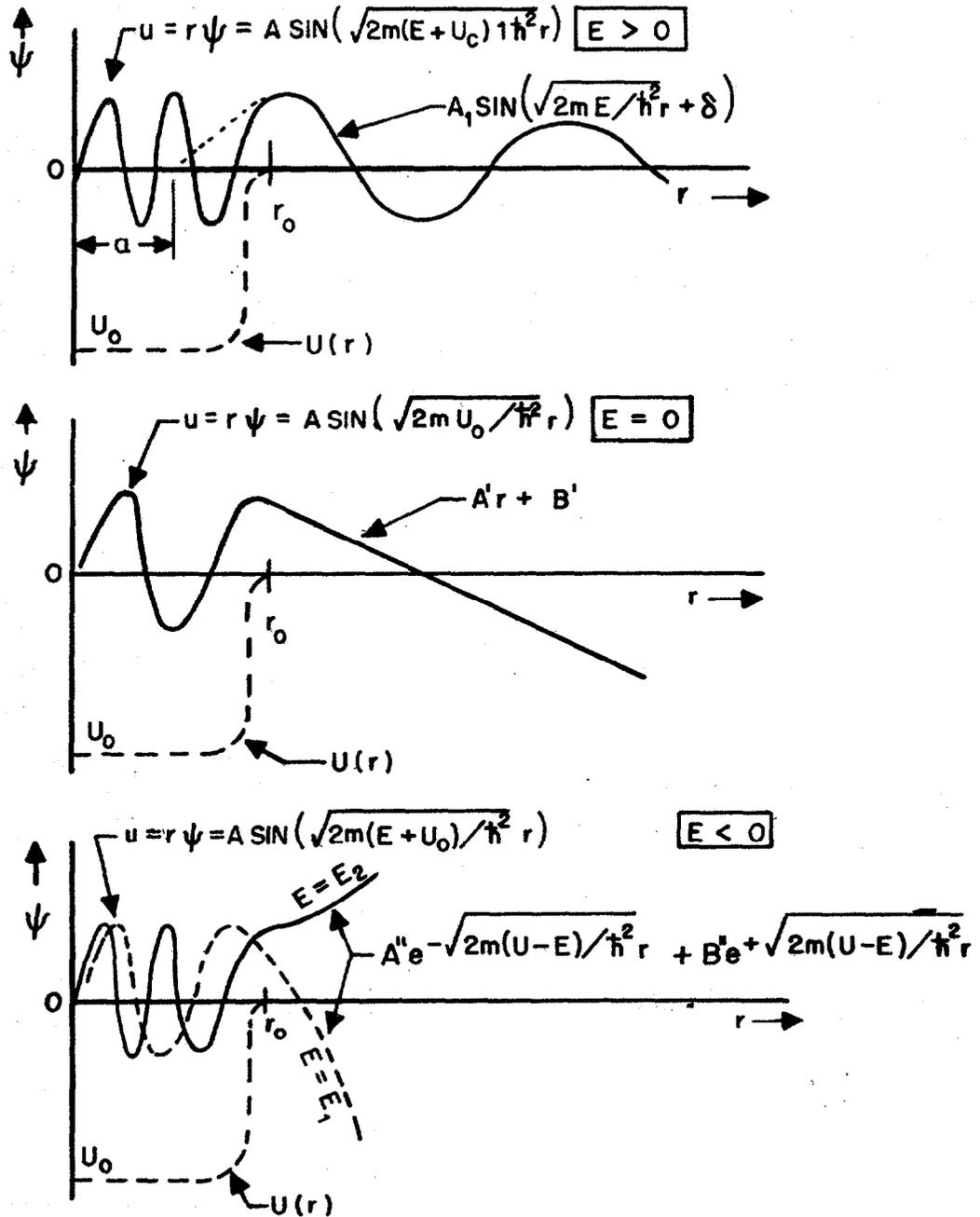


Figure 20. Scattering from a "potential well."

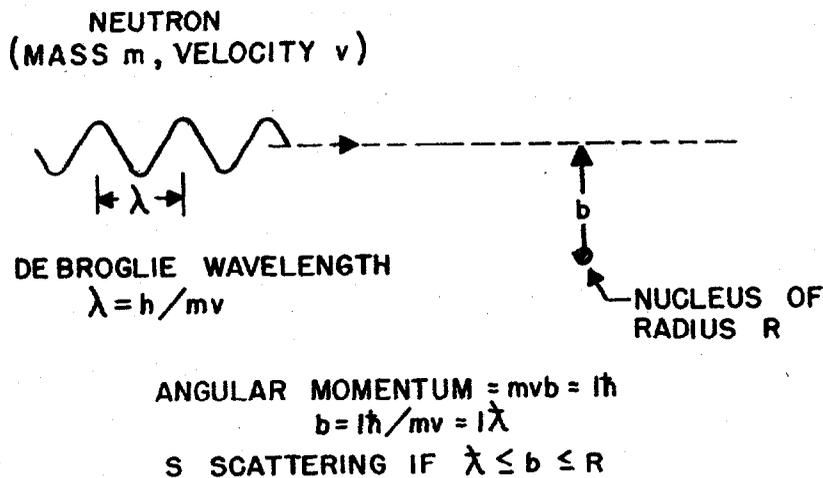


Figure 21. Quantitizing the angular momentum.

very small. Since $\delta = 2\pi(a/\lambda)$, we can replace $\sin^2 \delta$ by δ^2 and obtain

$$\sigma_{scat} = (4 \pi \hbar^2 / m^2 v^2) (4 \pi^2 a^2 / \lambda^2) = 4 \pi a^2 \quad (E \rightarrow 0) \tag{5-10}$$

The simplification follows from the relation $\lambda = h/mv$.

In the case of bound states, the scattering formula (5-9) can be shown to reduce to

$$\sigma_{scat} = \frac{4 \pi \hbar^2}{m [\epsilon + (E/2)]} \tag{5-11}$$

ϵ = binding energy of neutron when bound to the nucleus

E = kinetic energy of incident neutron

It is not always necessary for ϵ to be positive or "real." There are cases where "virtual" states of a nucleus may exist, these virtual states being characterized by negative ϵ . In such cases, the neutron wave function will be periodic inside and outside the nucleus but will have a larger amplitude inside than outside for that particular energy associated with the virtual state. When ϵ is negative, the absolute value is used in equation (5-11). An example of this occurs in the scattering of neutrons by protons. (The protons are considered "free" i.e., not bound chemically, in the following.) The cross section for scattering of slow neutrons by protons is dependent on what states are possible for the deuteron (combination of neutron and proton). There are two types of states, "singlet" and "triplet" states, associated with zero and unit total spin quantum number:

$S = 0$ (one singlet states; antiparallel spins)

$S = 1$ (three triplet states; parallel spins)

(Number of states = $2S + 1$)

It has been observed experimentally that the stable ground state of the deuteron is the $S = 1$ state, so that the triplet deuteron state has a positive ϵ . The $S = 0$ state is probably virtual, although

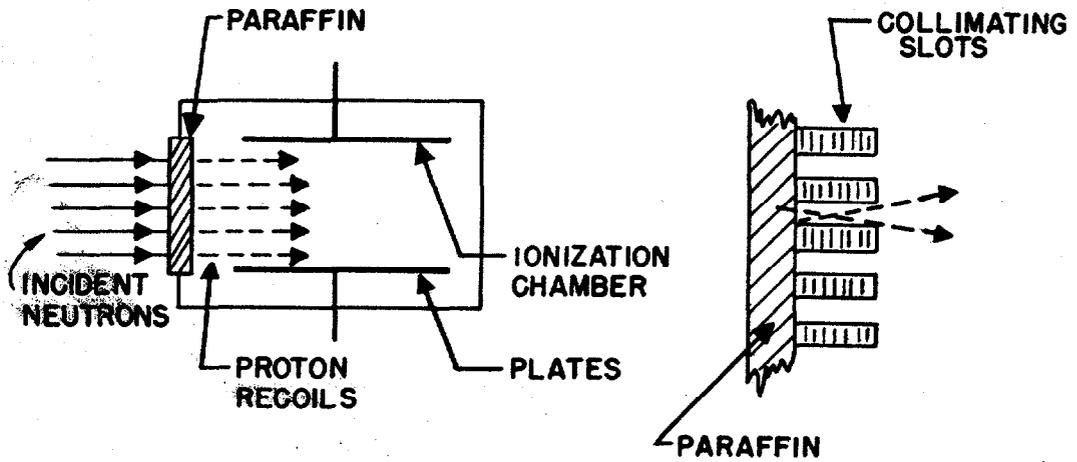


Figure 22. Fast neutron detector and recoil proton collimator.

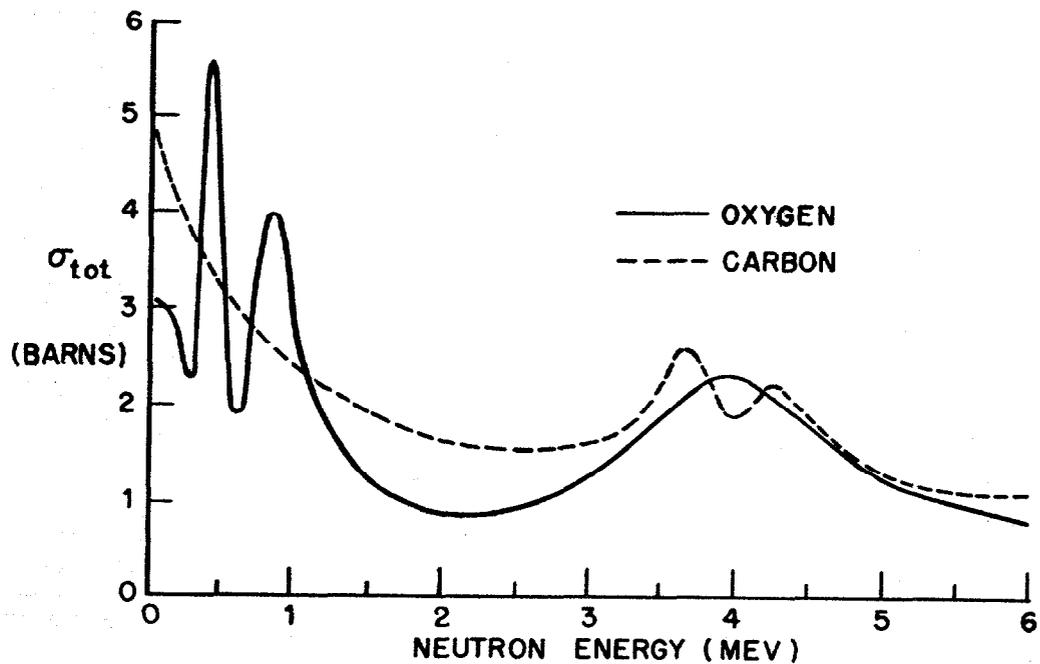


Figure 23. Carbon and oxygen cross sections. [Rev. Mod. Phys. 19:266 (1947).]

it is so small that it might be of either sign. Thus the scattering cross section for neutrons on free protons is:

$$\sigma_{\text{scat}} = \frac{4\pi\hbar^2}{M} \left[\frac{3}{4} \frac{1}{\epsilon_p + (E/2)} + \frac{1}{4} \frac{1}{|\epsilon_a| + (E/2)} \right]$$

where ϵ_p = energy of parallel spins state of the deuteron ϵ_a = energy of antiparallel spins state (virtual) and the 3/4 and 1/4 factors properly weight triplet and singlet states, respectively.

Direct experimental evidence for the existence of singlet and triplet states of the deuteron can be obtained from measurements of slow neutron scattering on hydrogen molecules. The hydrogen molecule can have two forms, "ortho" and "para." In orthohydrogen, the two protons have parallel spins, while in parahydrogen, they are antiparallel. Now a slow neutron (de Broglie wavelength larger than interatomic distance) incident on parahydrogen will be scattered by one proton with spin parallel to the incident neutron spin and by the other with antiparallel spin. The resultant neutron wave will be made up of real state scattering plus virtual state scattering (if the antiparallel spin, singlet state, of the deuteron is virtual). Since on theoretical grounds scattering for real states is 180° out of phase with that for virtual states, it follows that parahydrogen scattering should result in out-of-phase scattering from the two antiparallel spin hydrogen atoms, i.e., a small scattering cross section. Similarly, scattering of slow neutrons on orthohydrogen should result in scattering "in phase" from the two atoms, resulting in a larger scattering cross section. This has been observed experimentally and confirms the hypothesis of a virtual singlet state of the deuteron.

The scattering of neutrons in hydrogen is the basis of an important method for fast neutron detection. If a thin layer of paraffin is exposed to the neutron flux, then each neutron that is scattered gives rise to a recoil proton. The recoil proton has an energy of the same order of magnitude as the neutron energy. The proton, being charged, can be detected by an ionization chamber, as shown at the left in Figure 22. To measure neutron energies a collimating device (right in Figure 22) can be used so that only protons scattered in the direction of the incident neutron flux are allowed to enter the ionization chamber. The pulse in the chamber can be calibrated in terms of proton energy, and in this way the original neutron energies can be determined. Of course, the paraffin must be sufficiently thin to make multiple scattering of the incident neutrons negligible. (This limitation will be understood better after Chapter VI.)

In general, neutron scattering cross sections show complicated variations with energy. These irregularities are related to resonance phenomena not covered in the simple theory of these sections. Cross section versus neutron energy for two important scatterers, carbon and oxygen, are shown in Figure 23. It is not always possible to measure scattering cross sections directly. Generally, *total* cross sections are measured. For fast neutrons scattering predominates, so that the total cross section is effectively equivalent to the scattering cross section. For lower energies (below 1 Mev) this is not generally true, although for graphite and oxygen scattering predominates.

5.4 THE SCATTERING OF NEUTRONS

Referring to Figure 11, page 21, it is apparent that neutrons with energies less than 1 ev have de Broglie wavelengths of the order of Angstroms (10^{-8} cm) or greater. Since interatomic distances are likewise of the order of Angstroms, it might be expected that slow neutrons scattered by atoms will exhibit interference effects. This is actually the case. The anomalous scattering of slow neutrons by ortho- and parahydrogen, as we have seen, can be explained as an interference effect.

It is possible to study these effects by experiments analogous to those used in the study of x-ray diffraction and interference. Suppose a collimated flux of slow neutrons, Figure 24, is incident

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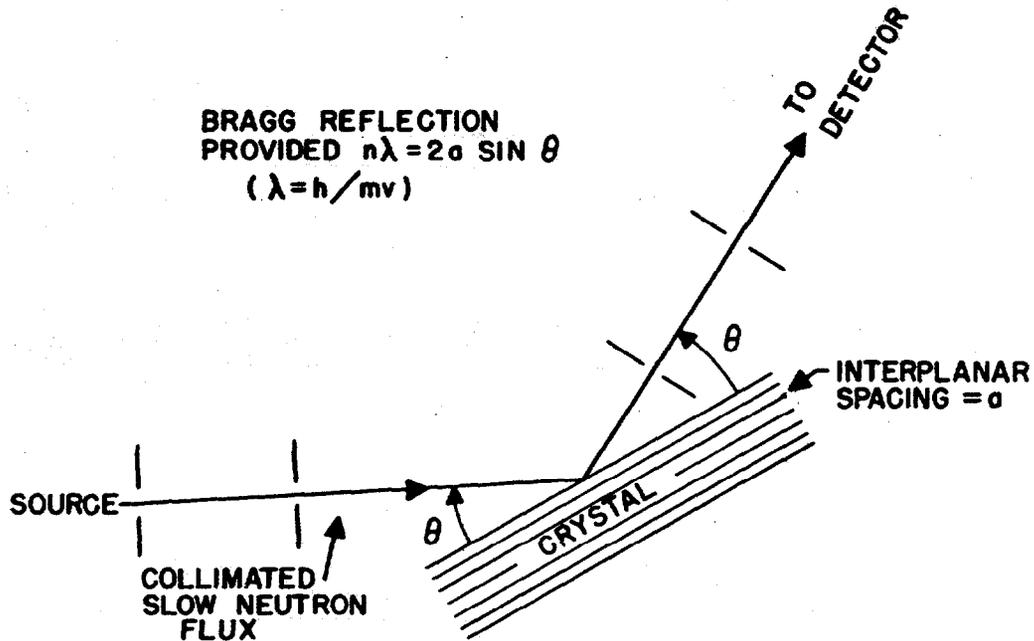


Figure 24. Scattering of neutrons by single crystal.

on a crystal. If the angle of incidence θ (angle between incident beam and crystal surface) is varied and for each value of θ the scattered intensity is measured with the detector, it is found that neutrons obey a Bragg-like formula:

$$n\lambda = 2a \sin \theta \quad (5-12)$$

where

n = order of interference

a = interplanar spacing

λ = de Broglie wavelength of neutron (h/mv)

For first order interference, equation (5-12) may be solved for the neutron velocity v :

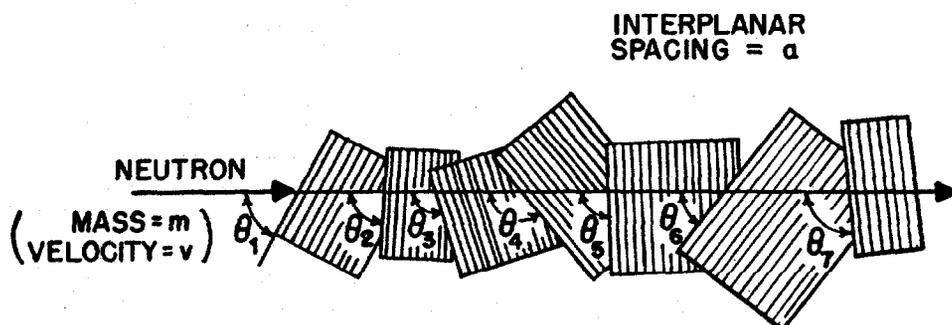
$$v = h/(2ma \sin \theta) \quad (5-13)$$

This shows that if a beam of neutrons with a continuous range of velocities impinges on a crystal, those neutrons of the proper v for the angle θ will be reflected in a sharp beam at an angle equal to the angle of incidence. Neutrons with other velocities will simply be scattered in the material in a normal way.

One can easily check the fact that the reflected beam really contains those neutrons whose velocity is given by the Bragg formula by taking readings with and without a boron absorber in front of the detector for various angles. The boron cross section as a function of v is well known (Figure 9, page 18), so that from the observed σ versus θ curve, one can compute v versus θ . This latter would be found to be the Bragg relation, aside from complications due to higher order reflections, etc.

It is apparent that the combination of a crystal and neutron detector can be used* to analyze a beam of neutrons for its velocity distribution (in a manner analogous to the analysis of an x-ray beam for wavelength distribution). For example, the slowed-down neutrons emerging from a tank

*Neutron crystal spectrometry has been made possible using the high neutron flux from chain reacting piles, see Phys. Rev. 70:557 (1946); 71:752, 757 (1947).



NEUTRON IS INCIDENT ON SUCCESSIVE MICRO
CRYSTALS AT ANGLES θ_1 θ_2 θ_3IF THE j th
MICROCRYSTAL IS SUCH THAT

$$v = h / (2ma \sin \theta_j)$$

THEN THE NEUTRON WILL BE SCATTERED

Figure 25. Scattering of neutrons by microcrystals.

of water, in which a neutron source has been placed, can be analyzed and found to have a Maxwellian distribution.

Microcrystalline substances scatter neutrons much better than regular crystalline substances. This can be understood by tracing the course of a neutron through a microcrystalline medium, Figure 25. If the Bragg condition is not fulfilled when the neutron arrives at the first microcrystal, then the neutron will pass on through. Otherwise, it will be reflected. When the neutron arrives at the next microcrystal, it must once again pass the test of not fulfilling the Bragg condition if it is not to be scattered. Were there but one crystal the neutron would have but one test to pass. However, with many randomly oriented microcrystals, the neutron has a large chance of being scattered. For a single large crystal, only those neutrons whose velocity satisfies the Bragg condition will be scattered. For the microcrystalline structure, sooner or later all velocity neutrons of the original beam will be removed as the beam passes from one crystal to the next.*

There is one very important difference between x-ray and neutron scattering. If the crystal is composed of two isotopes, the x-ray scattering is not particularly different from that which would be observed for a single isotope species, since x-ray scattering depends on the *extranuclear* properties of an atom. Since the extranuclear properties of two isotopes are very nearly the same, x-ray scattering is insensitive to isotope differences. On the other hand, in *neutron* scattering, the *nucleus* itself enters into the scattering process. Nuclei are such that in addition to determining the magnitude of the scattering cross sections, the *phases* of the scattered neutrons are determined.

Consider a neutron being scattered by two isotopes. If the first species has a scattering cross section σ_1 then the amplitude of the scattered neutron wave is proportional to $\sqrt{\sigma_1}$. Similarly, the second species scatters with amplitude proportional to $\sqrt{\sigma_2}$. Suppose we write these amplitudes in a sum and difference form:

*This effect was observed in early experiments on silica: Phys. Rev. 54:771 (1938); 55:1101 (1939). See also recent data in Phys. Rev. 70:815 (1946); 73:741 (1948).

$$\begin{aligned}\sqrt{\sigma_1} &= \left(\frac{\sqrt{\sigma_1} + \sqrt{\sigma_2}}{2} \right) + \left(\frac{\sqrt{\sigma_1} - \sqrt{\sigma_2}}{2} \right) \\ \sqrt{\sigma_2} &= \left(\frac{\sqrt{\sigma_1} + \sqrt{\sigma_2}}{2} \right) - \left(\frac{\sqrt{\sigma_1} - \sqrt{\sigma_2}}{2} \right)\end{aligned}\tag{5-14}$$

The first term is common to both, that is, the scattered amplitude is the same for this term (the so-called "coherent" part). The second term, "incoherent" part, is opposite in sign for the two amplitudes. The coherent part gives rise to interference, whereas the incoherent part gives rise to scattering as if from an unordered assembly of atoms.

How deep will a beam of neutrons penetrate inside a crystal if the Bragg condition is satisfied? Consider a beam of ν neutrons/cm²/sec incident on a simple cubic crystal, Figure 26. If there were but one atom in the crystal, then the number of neutrons scattered would be $\nu\sigma$ neutrons/sec. This would be isotropic, so that at a distance r from the crystal, the intensity would reduce to $\nu\sigma/(4\pi^2)$.

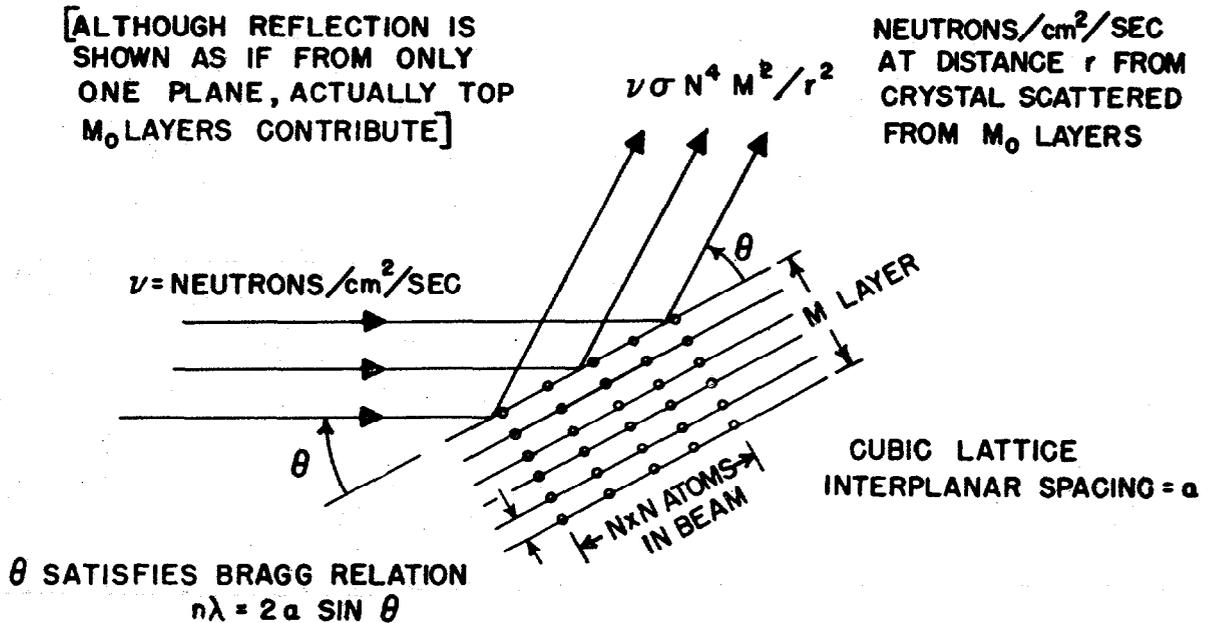


Figure 26. Bragg reflection of neutrons.

The amplitude of the neutron wave would be $\sqrt{\nu\sigma/4\pi}/r$. Let us estimate the amplitude of the scattered wave in the reinforced direction. This will be the amplitude for the single atom times the number of atoms participating in the scattering, that is

$$N \times N \times M \times \sqrt{\nu\sigma/4\pi} / r$$

The intensity in this direction will thus be $N^4 M^2 \nu\sigma / (4\pi r^2)$ neutrons/cm²/sec.

Now the angular spread* of the beam after scattering is just $\lambda /$ beam diameter or $\lambda / (Na \sin \theta)$. Thus the area of the scattered beam in the reinforced direction is r^2 times the angular width squared or $r^2 \lambda^2 / (Na \sin \theta)^2$. Multiplying this by the intensity gives the number of neutrons scattered in the reinforced direction:

$$\begin{aligned} & [N^4 M^2 \nu\sigma / (4\pi r^2)] [r^2 \lambda^2 / (Na \sin \theta)^2] \\ & = \nu\sigma N^2 M^2 (\lambda/a)^2 / (4\pi \sin^2 \theta) \text{ neutrons/sec} \end{aligned}$$

It is to be noted that in the above derivation, the attenuation of the neutron intensity in passing through the M layers has been neglected. If we further simplify by considering all reflections are first order ($n=1=2a \sin \theta/\lambda$), then the total number of neutrons per second in the reinforced direction is

$$\text{Neutrons/sec scattered} = \nu\sigma N^2 M^2 / \pi \tag{5-15}$$

Now if the crystal were a perfect reflector, then all incident neutrons would be removed from the beam, that is, just ν (neutrons/cm²/sec) times $(Na \sin \theta)^2$, the beam area. This number can be considered the upper limit for the scattered beam. Thus

$$\nu\sigma N^2 M^2 / \pi < \nu N^2 a^2 \sin^2 \theta$$

or
$$M^2 < a^2 (\pi/4) (\lambda/a)^2 \tag{5-16}$$

the latter since $\sin \theta \approx \lambda/2a$ for first order effects. If we assume $\lambda \approx a$, the inequality may be written as an order of magnitude relation $M < a/\sqrt{\sigma}$.

What is the significance of this inequality? It means that the layers beyond $M_0 = a/\sqrt{\sigma}$ do not contribute to the scattered beam. Put another way, the beam does not penetrate beyond M_0 layers. Numerically, if a is 3×10^{-8} cm and σ about 4×10^{-24} cm²/atom, then M_0 is of the order of $3 \times 10^{-8} / 2 \times 10^{-12} \approx 10^4$. Thus about 10^4 planes play a vital part in Bragg reflection. The depth of penetration will be $M_0 a$ or $\sim 5 \times 10^{-4}$ cm = 5 microns.

Throughout the foregoing discussion, it has been assumed that the only attenuation of neutrons is due to Bragg-like reflection. Consider a heterogeneous beam of neutrons incident on a large perfect crystal. Those neutrons that satisfy the Bragg condition will be scattered out of the beam in penetrating the first few microns. However, those that do not satisfy the Bragg condition generally will not be transmitted without some loss in intensity. The reasons for this are:

1. The presence of isotopes makes a random irregularity resulting in incoherent scattering for all velocities.
2. The random variation of the spin direction of nuclei also results in incoherent scattering.
3. Even if the crystal were regular, the thermal motions of the atoms would contribute to non-Bragg scattering.
4. The crystal atoms will generally have a finite absorption cross section even though $\sigma_a \ll \sigma_s$.

*The formula should be multiplied by a constant depending on shape of beam area. See any standard physical optics text.

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It is possible to use interference phenomena to obtain very slow neutrons. Rewriting the Bragg formula, equation (5-12), as $\lambda = 2a (\sin \theta/n)$ or $\lambda < 2a$, it is apparent that neutrons with $\lambda > 2a$ cannot be Bragg-reflected. All scattering of neutrons with wavelength greater than twice the interplanar spacing would be due to the four listed causes. If graphite is used, the fact that there is only one isotope (~99%) and the spin is zero (even mass numbers usually have zero spins) minimizes scattering from spin and isotopic irregularities. For graphite, $2a = 6.69 \times 10^{-8}$ cm, so that the limiting λ is 6.69 Angstroms, corresponding to a neutron energy of $(0.2848)^2/\lambda^2$ (see Figure 11, page 21), or 0.0018 ev. Suppose (Figure 27) a Maxwellian distribution of neutrons, peaking at 0.025 ev for graphite at room temperature, is incident on a polycrystalline piece of graphite. Then, as a consequence of interference, the graphite will (in an appreciable distance) weed out all neutrons with energies above 0.0018 ev. Only those very slow neutrons with $\lambda > 2a$ will not be scattered out of the beam by Bragg reflections. Moreover, because of relative isotopic uniformity, zero spin, and small absorption, these "cold" neutrons will be able to get through the graphite with minimum attenuation. Thus it is possible to secure a very "cold" beam using thermal neutrons and a polycrystalline graphite "filter."*

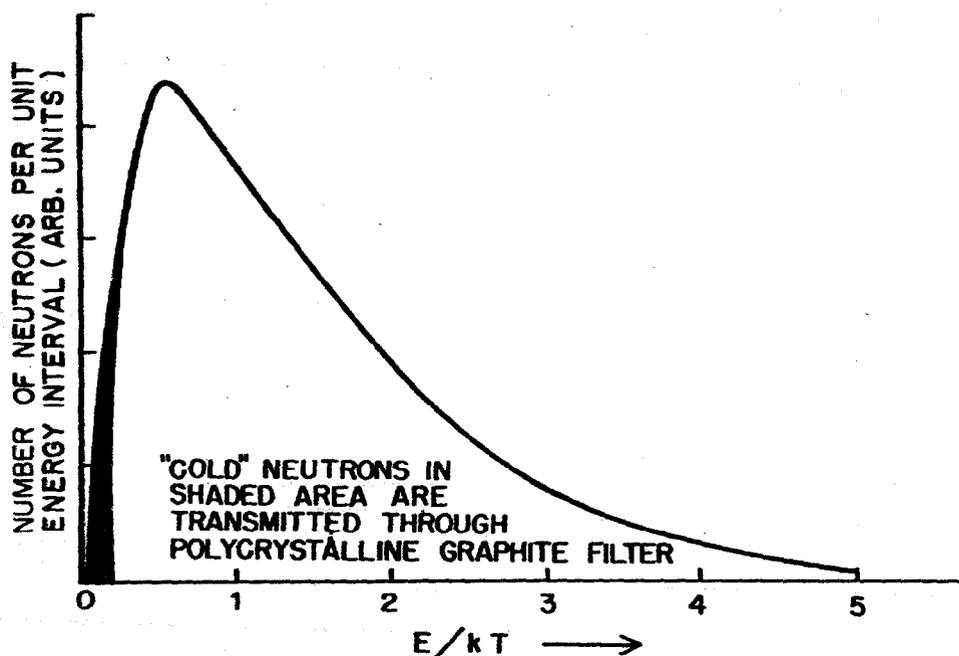


Figure 27. Maxwell distribution and "cold" neutrons.

*See "Production of Low Energy Neutrons by Filtering through Graphite" by H. L. Anderson, E. Fermi, L. Marshall, *Physical Review* 70:815 (1946). With a 23 cm graphite filter, neutrons corresponding to 18K (or 7.15 Angstroms) were obtained.

Neutrons exhibit total reflection characteristics in a manner analogous to x radiation. If a beam of x rays is incident on a polished surface at a glancing angle, the beam is totally reflected. Most substances have indices of refraction for x rays very slightly less than unity. The index of refraction is closely related to the scattering properties of the substances since the interference of scattered x rays and incident x rays is responsible for the resulting wave transmitted in the substances. The change of phase in the transmitted wave can be described most conveniently in terms of wave velocity change or a refractive index. The same phenomena occur for neutrons incident at glancing angles on a polished surface. The index of refraction for neutrons is also very close to unity. This means that a converging lens for neutrons would have to bulge very much along the axis to be effective if it were made of substances where the refractive index is slightly greater than unity. For substances in which the refractive index is less than unity, a converging lens would look like the diverging lens of optics, that is, very thin close to the axis and thick at distances far from the axis. While these lenses are possible in principle, the fact that the refractive index is always very close to unity makes neutron lenses impractical.

PROBLEMS

1. Given $R = 0.282 \times 10^{-12}$ cm (the classical electron radius) and a depth of a rectangular potential well of 10.8 Mev (the singlet state of the deuteron) and of 19.7 Mev (triplet state), answer the following questions concerning the scattering of neutrons by protons.

1. Are there any bound states in each of these cases?
2. What is the value of "a"?
3. What is the average cross section for low velocity neutrons in hydrogen?

2. For NaCl [using the (001) planes only] make a table of the λ reflected (1st and 2nd order) at the following various values of angle $\theta = 1, 2, 3, 4, 5, 10, 20, 30, 40^\circ$. Calculate the neutron energy in ev for each λ . What is the relative intensity of first and second order beams, assuming a Maxwell distribution ($T = 300^\circ\text{K}$) for the neutrons?

3. Show that if a continuous distribution of neutrons impinges on a microcrystalline substance where $M < M_0$, that the scattered intensity is of the order of that expected from a noncrystalline substance of the same number of atoms for all energy neutrons. Use the fact that the resolution of a microcrystal reflecting neutrons of wavelength λ according to the Bragg formula is given by $\delta\lambda/\lambda \sim 1/M$.

CHAPTER VI

THE SLOWING DOWN OF NEUTRONS

6.1 THE CHANGE OF DIRECTION AND ENERGY UPON COLLISION

When a neutron is elastically scattered by a nucleus, the nucleus being initially at rest, generally there is a transfer of kinetic energy from the neutron to the nucleus. If the struck nucleus is hydrogen, then the neutron will lose of the order of half its energy in the collision. Successive collisions will, on the average, continue this process, so that a 1 Mev neutron becomes thermal (0.025 ev) in about 24 collisions.

In this chapter, we shall consider in detail the nature of this slowing down process. It will be found that classical collision theory is applicable and leads to results in agreement with experiment. As in the classical problems of colliding bodies, it is convenient to set up two frames of reference:

The laboratory system (R system) - In this system, the frame of reference is determined by considering the target body at rest before the collision.

The center of gravity system (C system) - In this the frame of reference is determined by considering the center of gravity of target and projectile at rest.

In the latter, it is apparent that we are looking at the assemblage of colliding bodies as a whole and considering its center of gravity as stationary. It is important to note that all *experimental measurements* made of nuclear collisions use the *laboratory system* of reference. On the other hand, practically all *laboratory system* of reference. On the other hand, practically all *theoretical calculations* are made in the *center of gravity system*. We shall see that the C system affords a view of things that is basically simpler than that of the R system.

Consider in the R system the collision of a neutron (mass = 1) of velocity v with a nucleus of mass A , initially at rest. Since the total mass of the system of colliding particles is $A + 1$ and the initial neutron momentum is $1 \cdot v$, it follows that the velocity of the center of gravity (as seen in the R system) is $v/(A + 1)$. The velocities of the nucleus A and the neutron relative to the center of gravity are $v/(A + 1)$ and $vA/(A + 1)$, respectively. These velocities are in opposite directions, so that the total momentum of the system with respect to the center of gravity is zero. After the collision, the magnitudes of the velocities are unchanged, but the directions of motion are along a different line. (see Figure 28). The magnitudes must remain unchanged, since the total momentum with respect to the center of gravity must remain zero. The change in direction will depend upon the exact nature of the collision. If θ is the angle between the initial direction and scattered direction (in the C system), then $\theta = 0$ means the collision was "glancing" while $\theta = \pi$ means it was "head-on".

It is apparent that an observer in the C system would see the two colliding bodies initially heading for each other along a single straight line, with the heavier body moving more slowly. After the collision, the C system observer would see the two bodies moving away in diametrically opposite directions, with unchanged velocities. The C system observer would say that the collision process was *isotropic* if all angles (θ) between "before" and "after" directions of motion were equally probable. As the C system has no preferred direction of motion (center of gravity at rest), we adopt the same terminology and call the *scattering isotropic* when all values of θ are equally probable.

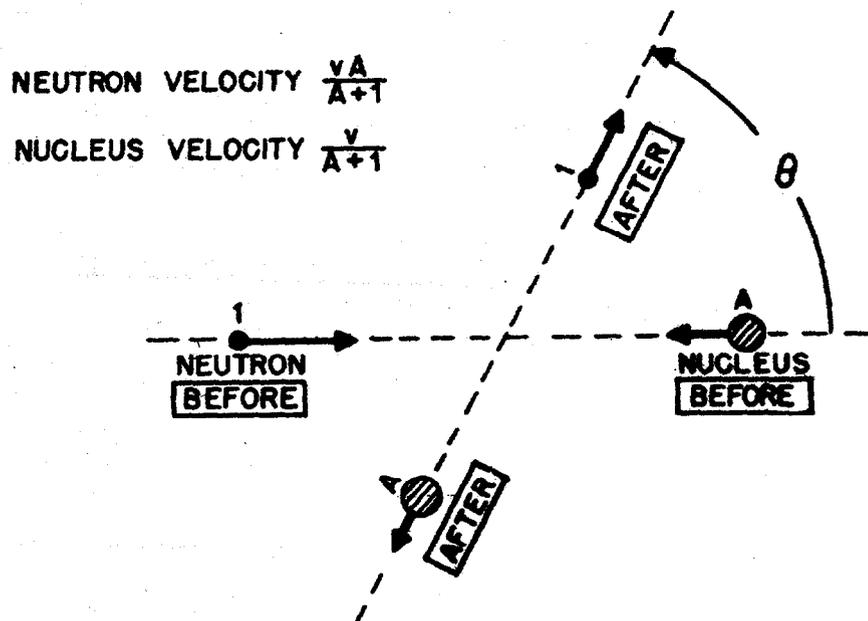


Figure 28. Neutron Scattering: Center of gravity system.

Now the transformation from the C system to the R system is readily made if we consider that the C and R systems move relative to each other with a velocity equal to the velocity of the center of mass in the laboratory system. This velocity is the same as the initial velocity of the nucleus A in the C system, or just $v/(A + 1)$. Taking this vector and adding it to the "after" velocity of the neutron in the C system [$vA/(A + 1)$] gives us the "after" velocity (v') of the neutron in the R system (see Figure 29). It is apparent that the angle of scattering in the C system, that is, θ , is not the same as the angle of scattering in the R system, ϕ . It is readily proved (by the sine law) that

$$A \sin(\theta - \phi) = \sin \phi$$

or

$$\tan \phi = \frac{A \sin \theta}{1 + A \cos \theta} \tag{6-1}$$

Both expressions are equivalent. With either, it is possible to convert scattering angles from one system to the other.

The neutron energy before the collision in the R system is $mv^2/2$. After the collision, it is $mv'^2/2$. Thus the ratio of neutron energies after and before collision as observed in the R system is $(v'/v)^2$. This ratio can be determined from Figure 29 by the cosine law:

$$v'^2 = [v/(A + 1)]^2 + [vA/(A + 1)]^2 + [2Av^2/(A + 1)^2] \cos \theta \tag{6-2}$$

$$\frac{\text{K.E. after collision}}{\text{K.E. before collision}} = \frac{E'}{E} = \frac{A^2 + 1 + 2A \cos \theta}{(A + 1)^2}$$

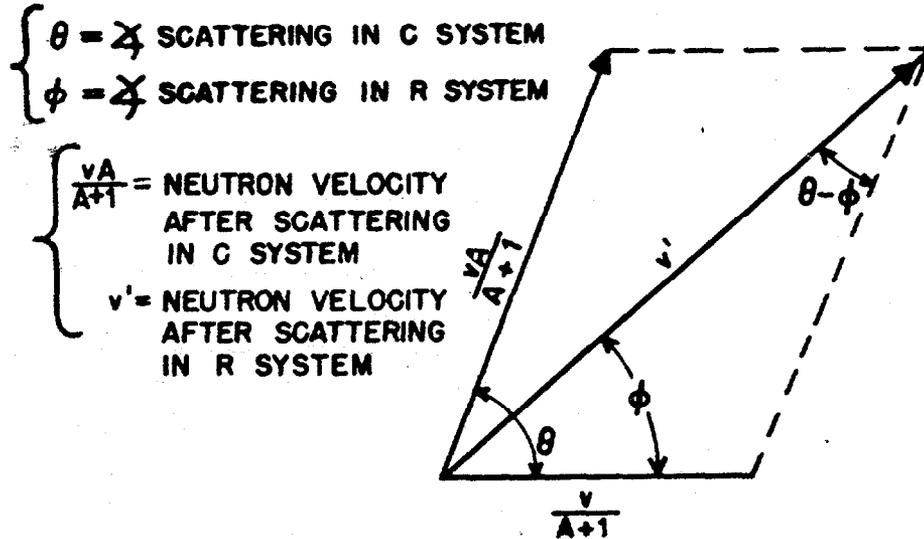


Figure 29. Neutron Scattering in laboratory (R) system.

It should be noted that although equation (6-2) is the ratio of the kinetic energies observed in the laboratory system, nevertheless, the angle θ is the angle of scattering as observed in the center of gravity system. This could be expressed in terms of the laboratory scattering angle ϕ by means of equation (6-1). However, since we are going to want to average over all angles, we will keep the C system scattering angle, as averaging in the C system is relatively simple. In fact, for isotropic scattering, the average of the cosines of θ is zero:

$$(\cos \theta)_{av} = \left[\int_0^\pi \cos \theta \cdot 2\pi \sin \theta \, d\theta \right] / \left[\int_0^\pi 2\pi \sin \theta \, d\theta \right] = 0 \quad (6-3)$$

On the other hand, the average of the cosine of the laboratory scattering angle is not equal to zero but is a positive number, showing that the colliding particles have a tendency to preserve their direction of motion. From equation (6-1), it follows that:

$$\cos \phi = 1 / \sqrt{1 + \tan^2 \phi} = \frac{A \cos \theta + 1}{\sqrt{A^2 + 1 + 2A \cos \theta}}$$

Averaging the $\cos \phi$ as before:

$$(\cos \phi)_{av} = \frac{\int_0^\pi \cos \phi \cdot 2\pi \sin \theta \, d\theta}{\int_0^\pi 2\pi \sin \theta \, d\theta} = \frac{\int_0^\pi (A \cos \theta + 1) \sin \theta \, d\theta}{2(A^2 + 1 + 2A \cos \theta)^{3/2}} = \frac{2}{3A} \quad (6-4)$$

As was expected, the average of the cosine is greatest for neutron collisions with lighter nuclei; that is, the tendency to keep going in the original direction is greatest when the target nucleus has the least mass.

Let us re-examine the ratio of neutron kinetic energies after and before a collision, E'/E of equation (6-2). The maximum and minimum values are:

$$\frac{E'}{E} = \frac{A^2 + 1 + 2A \cos \theta}{(A + 1)^2} = \begin{cases} 1 \text{ for } \theta = 0 \text{ (glancing collision)} \\ \left(\frac{A-1}{A+1}\right)^2 \text{ for } \theta = \pi \text{ (head on-collision)} \end{cases} \quad (6-5)$$

In collisions with hydrogen, $A = 1$, the limits are thus zero and unity. For heavier atoms, it is impossible to bring the neutron to rest. When $A \gg 1$, the minimum is $(A-1)^2/(A+1)^2 = 1 - (4/A) + (8/A^2) \dots$, so that for $A = 100$, the greatest possible loss of neutron kinetic energy after a single scattering is 4%. For $A = 200$, it is 2%.

Now what is the relative probability of a neutron's having an energy E' (between the limits just described) after a collision? Assuming isotropic scattering, the probability dp that the neutron is scattered into the solid angle between θ and $\theta + d\theta$ is equal to that solid angle divided by the total solid angle 4π :

$$dp = (\text{Solid angle between } \theta \text{ and } \theta + d\theta) / 4\pi = 2\pi \sin \theta d\theta / 4\pi = \sin \theta d\theta / 2$$

The relation between dp and the range of final energy which corresponds to this range of angles is found by differentiation of equation (6-2).

$$dE' = - \frac{2AE \sin \theta d\theta}{(A+1)^2} = -E \frac{4A}{(A+1)^2} dp$$

The negative sign means that increasing θ corresponds to decreasing E . Hence

$$\left\{ \begin{array}{l} \text{The probability that the} \\ \text{final energy is between} \\ E' \text{ and } E' + dE' \end{array} \right\} = dp = \frac{(A+1)^2}{4A} \frac{dE'}{E} \quad (6-6)$$

where the negative sign has been dropped so dE' is considered positive. Equation (6-6) means that the probability of the final energy being E' is independent of E' . Figure 30 is a graph of the distribution function $p(E')$ versus E'/E , showing it to be a constant, $(A+1)^2/4A$, between $E'/E = (A-1)^2/(A+1)^2$ and $E'/E = 1$. As a check on the normalization of (6-6), it should be noted that the area under the distribution curve is unity:

$$\int p(E') dE' = \int_{\left(\frac{A-1}{A+1}\right)^2}^1 [(A+1)^2/4A] d(E'/E) \\ = [(A+1)^2/4A] \left[1 - \left(\frac{A-1}{A+1}\right)^2 \right] = 1$$

Using this distribution function, we could now calculate the average of E'/E . However, it is more convenient to consider the average of the natural logarithm of the energy ratio that is, $\log_e (E/E')$. This is due to the fact that, since the per cent loss in energy is on the average the same, the neutron's energy decreases in successive collisions, as shown in Figure 31. In each collision, it is the $\log_e E$ rather than E which changes by a relatively fixed amount. Evaluation of the average of $\log_e (E/E')$ proceeds as follows:

$$\zeta = \left[\log_e (E/E') \right]_{av} = \int_{\text{Min } E'}^{\text{Max } E'} \log_e (E/E') p(E') dE' \\ = \int_{\frac{E(A-1)^2}{(A+1)^2}}^E \log_e (E/E') [(A+1)^2/4A] \frac{dE'}{E}$$

After some algebra, this reduces to:

$$\zeta = 1 - \frac{(A-1)^2}{2A} \log_e \frac{A+1}{A-1}$$

For $A \gg 1$, this can be reduced to $\zeta \sim (2A - \frac{4}{3})/A^2$ or $\zeta \sim 2/(A + \frac{2}{3})$.

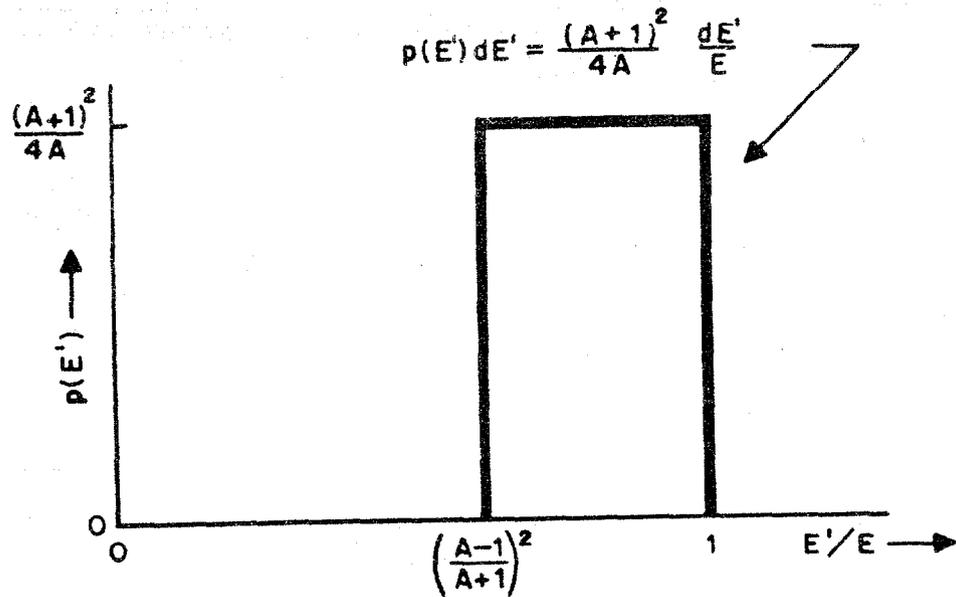


Figure 30. Neutron energy distribution after single elastic scattering.

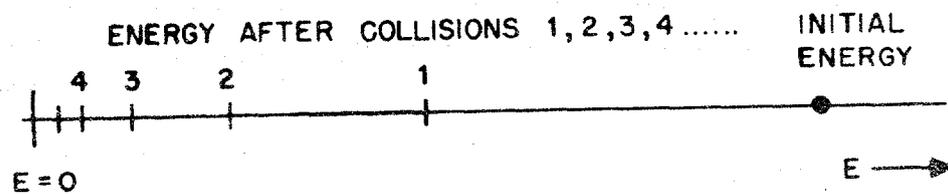


Figure 31. Neutron energy after successive collisions.

The average logarithmic energy decrease per collision, ζ , is 0.158 for carbon ($A=12$). For hydrogen, $\zeta = 1 = \text{average } \log_e E/E'$ so that E'/E is, on the average, $1/e$. To reduce a 1-Mev neutron to 0.025 ev (thermal energies) by collisions with hydrogen, one would require ν collisions with:

$$\nu = \frac{\log_e (E/E')}{\log_e (E/E')_{\text{av}}} = \frac{\log_e (10^6/0.025)}{\zeta} = \log_e 4 \cdot 10^7 = 17.5$$

For carbon, since ζ is 0.158, about $17.5/0.158$ or 110 collisions would be required.

6.2 NEUTRON DISTRIBUTION FROM POINT SOURCE - EXPERIMENTAL METHODS

Knowledge of the nature of the process of slowing down of neutrons by collisions is essential in the treatment of most problems in which neutron fluxes are introduced into media. Perhaps the simplest question to be answered is: "Given a point source of monoenergetic neutrons, what is the steady state spatial distribution as a function of energy?" The answer will be basic, since any source distribution can be considered a superposition of point sources.

Suppose there is a radium-beryllium neutron source in a large tank of water. For hydrogen the scattering cross section is particularly large at low energies so that a 1-Mev neutron will do most of its traveling between the first few collisions, as shown qualitatively in Figure 32. The distribution of neutrons from the source in water can be investigated by using detectors sensitive to different energy neutrons. Materials such as indium, rhodium, or iodine, each of which has a strong resonance for one particular neutron energy, can be used as detectors for that energy, provided that the neutron absorption corresponding to the resonance level results in the production of radioactivity. Figure 33 sketches the cross section versus energy curve for indium. The reference given with the figure contains further details and bibliographies on activation of such materials. When possible, the detectors are made into foils. Elimination of the effects of thermal neutrons (since most of the detectors, in addition to being responsive to particular resonance energy neutrons, are generally responsive to thermal neutrons as well) is accomplished by surrounding the detector with cadmium. The effect of cadmium may be seen by inspection of its cross section versus energy curve, the broken line of Figure 33.

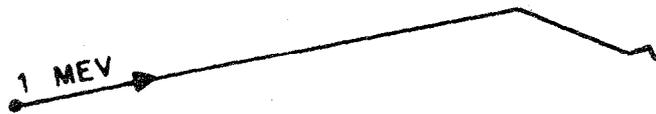


Figure 32. Mean free path decreases as energy does in hydrogenous materials.

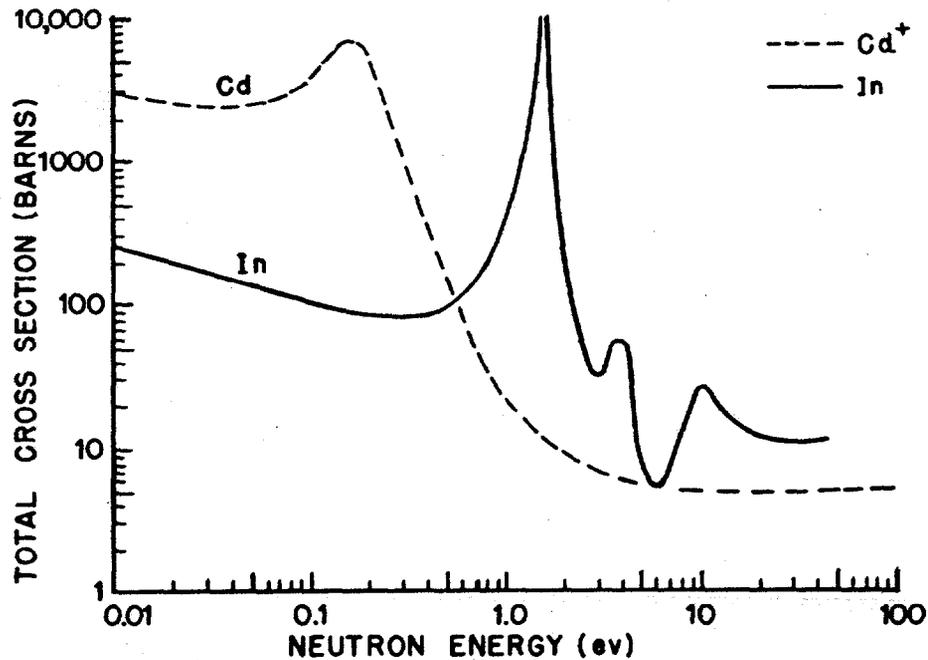


Figure 33. Cross section versus energy curve for indium.

Returning to the neutron source in the tank of water, it is apparent that the investigation of the distribution of 1.44-eV neutrons (indium resonance) may be made* by placing indium foils sandwiched between cadmium foils at various positions in the tank. The degree of activation of the indium (taking into the account the decay of radioactive indium during the period of exposure) in the various positions is proportional to the density of 1.44-eV neutrons at those positions. If the cadmium foils were removed and the difference between the activation of the bare indium foil and the activation of the sandwiched foils (Cd-In-Cd) were computed, then the relative densities of thermal (cadmium) neutrons can be determined for the various tank positions. The spatial distribution of ≈ 37 -eV neutrons can be determined using an iodine detector.† Putting all these curves together yields the radial distribution of the density of the various energy neutrons in the water, as sketched on Figure 34.

* For details see "Ra- α -Be Neutrons in Water" by J. N. Rush in Phys. Rev. 73:271(1948).

† For iodine resonance see Phys. Rev. 71:174(1947). There are apparently two resonance levels, ≈ 32 eV and ≈ 42 eV.

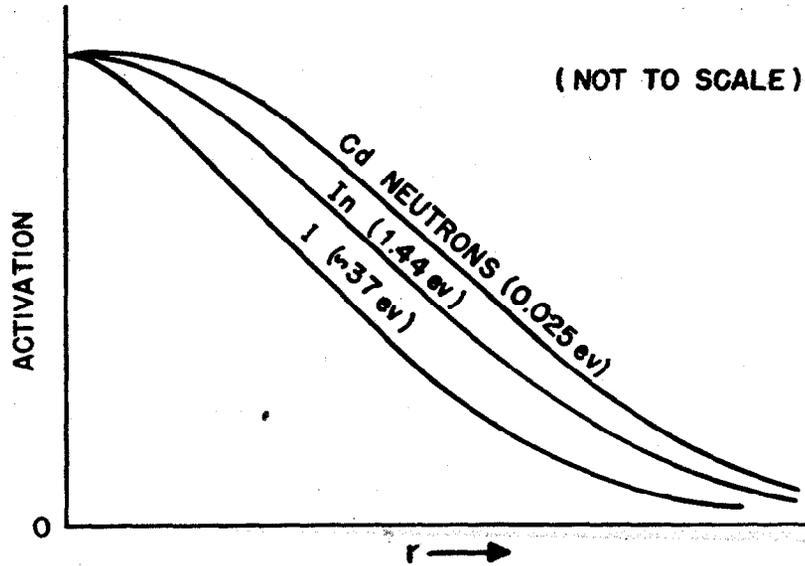


Figure 34. Neutron source in water.

6.3 DISTRIBUTION OF NEUTRONS FROM POINT SOURCE - CALCULATION OF r_{av}^2

Consider a point source of neutrons of energy E_0 located in an infinite homogeneous medium. The neutrons are slowed down by collisions after leaving the source. Consider all the neutrons of energy E . How far away from the source are they on the average? What is the average of the square of the distance from the source r_{av}^2 ?

If a neutron from a source S undergoes successive collisions (Figure 35), with successive displacements of $\vec{l}_1, \vec{l}_2, \vec{l}_3, \vec{l}_4, \dots, \vec{l}_n$, then the resultant displacement is:

$$\vec{r} = \vec{l}_1 + \vec{l}_2 + \vec{l}_3 + \dots + \vec{l}_n$$

and the square of the displacement ($\vec{r} \cdot \vec{r}$) is:

$$r^2 = l_1^2 + l_2^2 + l_3^2 + \dots + l_n^2 + 2(\vec{l}_1 \cdot \vec{l}_2 + \vec{l}_1 \cdot \vec{l}_3 + \dots + \vec{l}_2 \cdot \vec{l}_3 + \dots)$$

To obtain the average of the foregoing expression, we break the averaging into three steps (Figure 36). First, we average over the azimuthal angle ψ keeping ϕ , the angle of scattering in the laboratory system, and the lengths l constant. Then we average over l and finally over ϕ . The average over ψ is accomplished by use of the theorem:

$$[\cos(1, 4)]_{\psi_{av}} = \cos(1, 2) \cos(2, 3) \cos(3, 4) \tag{6-8}$$

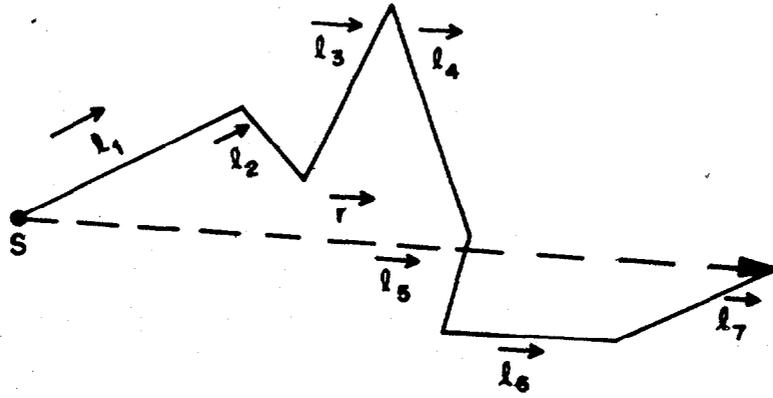


Figure 35. Typical neutron paths.

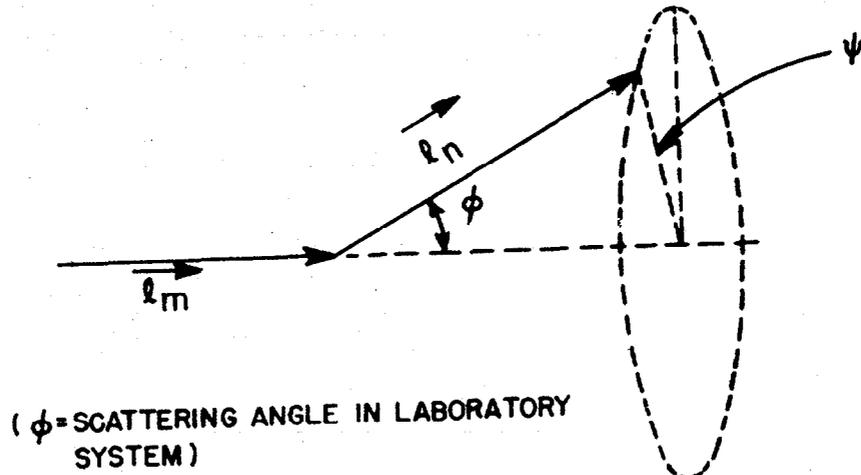


Figure 36. Displacement vectors in neutron scattering.

with $\cos(m,n) = \text{cosine of angle between } \vec{l}_m \text{ and } \vec{l}_n$

This theorem may be proved by considering \vec{l}_1 fixed. Then, with \vec{l}_2 fixed, consider the vector \vec{l}_4 to be averaged by allowing \vec{l}_4 to rotate around \vec{l}_3 . Figure 37 shows the three vectors \vec{l}_1 , \vec{l}_3 , and \vec{l}_4 arranged to form edges of a tetrahedron. The base plane has been constructed by passing a plane perpendicular to \vec{l}_3 and intersecting \vec{l}_1 at unit length. The figure is to be used in the averaging process where \vec{l}_4 is allowed to rotate about \vec{l}_3 , so that the angle ψ indicated on the diagram is to be varied from 0 to 2π in the averaging process. Considering the triangles APB and BPC of Figure 37 it is apparent from elementary trigonometry that

$$AP = 1; AB = \sin(1,3); PB = \cos(1,3); BC = PB \cdot \tan(3,4) = \cos(1,3) \tan(3,4)$$

$$PC = PB / \cos(3,4) = \cos(1,3) / \cos(3,4)$$

The quantity $(AC)^2$ can be found by use of the cosine law in triangle APC and in triangle ABC. Equating these two values gives:

$$(AC)^2 = (AP)^2 + (PC)^2 - 2(AP)(PC) \cos(1,4) = (AB)^2 + (BC)^2 - 2(AB)(BC) \cos \psi$$

Solving this equation for $\cos(1,4)$ and substituting the trigonometric formulas for the various sides (AP, PC, AB, BC) yield the following:

$$\cos(1,4) = \cos(1,3) \cos(3,4) + \sin(1,3) \sin(3,4) \cos \psi$$

As \vec{l}_4 rotates around \vec{l}_3 the $\cos \psi$ averages out to zero. The angle between these two vectors (3,4) being constant during this averaging process means that the average of the $\cos(1,4)$ becomes:

$$[\cos(1,4)]_{av} = [\cos(1,3)]_{av} \cdot \cos(3,4) \tag{6-9}$$

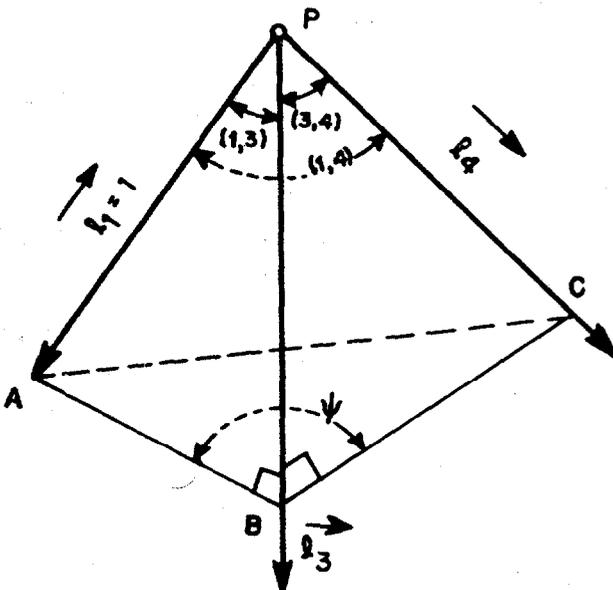


Figure 37. Averaging over azimuthal angle (ψ).

Now, in the above, the vector \vec{l}_2 was kept fixed and \vec{l}_4 averaged. If \vec{l}_4 is kept fixed and \vec{l}_3 allowed to rotate about \vec{l}_2 we find, in a manner similar to that used in deriving (6-9), that:

$$[\cos(1,3)]_{av} = [\cos(1,2)]_{av} \cdot \cos(2,3) \quad (6-10)$$

However, since vectors \vec{l}_1 and \vec{l}_2 are successive vectors the average of the $\cos(1,2)$ is a constant; that is, when \vec{l}_2 rotates about \vec{l}_1 , the angle (1,2) remains constant. Substitution of equation (6-10) into equation (6-9) proves the theorem stated in equation (6-8). The theorem may be generalized to apply to any succession of such vectors averaged over angle ψ (as shown in Figure 36):

$$[\cos(a,z)]_{\psi av} = \cos(a,b) \cdot \cos(b,c) \cdot \cos(b,c) \dots \cos(y,z) \quad (6-11)$$

($\vec{l}_a, \vec{l}_b, \vec{l}_c, \dots, \vec{l}_z$ are successive vectors)

Now consider the average over l . If λ_n is the mean free path, then the probability that l_n lies between l_n and $l_n + dl_n$ is $\exp(-l_n/\lambda_n)$. The average of l_n and l_n^2 can be readily determined:

$$(l)_{av} = \left[\int_0^{\infty} l \exp(-l/\lambda) dl \right] / \left[\int_0^{\infty} \exp(-l/\lambda) dl \right] = \lambda \quad (6-12)$$

$$(l^2)_{av} = \left[\int_0^{\infty} l^2 \exp(-l/\lambda) dl \right] / \left[\int_0^{\infty} \exp(-l/\lambda) dl \right] = 2\lambda^2$$

(The subscripts "n" have been omitted for simplification of notation.) The first relation of equation (6-12) is in a sense a definition, since the average of l is just the mean free path.

Returning to the original equation for r^2 , we may now write the equation for r^2 averaged over the ψ 's and l 's:

$$\begin{aligned} r^2_{l,\psi av} &= 2\lambda_1^2 + 2\lambda_2^2 + 2\lambda_3^2 + \dots + 2\lambda_n^2 + 2[\lambda_1 \lambda_2 \cos(1,2) \\ &+ \lambda_1 \lambda_3 \cos(1,2) \cos(2,3) + \dots + \lambda_2 \lambda_3 \cos(2,3) + \dots] \end{aligned} \quad (6-13)$$

In the case of hydrogen, this calculation can be completed in an exact way. Generally it is convenient to make some approximations. Let us assume that it takes a large number of collisions to produce a small change in energy. This will be valid for neutron collisions with heavy atoms ($A \gg 1$). The various angles of equation (6-13), (1,2), (2,3), etc., may take on all values from 0 to π . We must average over the various possible angles, recalling that these angles are the successive angles of scattering as measured in the laboratory system, angle ϕ of Figure 29. As we have proved in equation (6-4), the average of the cosines of these angles for isotropic scatterers is $2/(3A)$, a number that decreases with increasing A . This fact may be used to advantage if we note the coefficient of λ_1 in (6-13):

$$2[\lambda_1 + \lambda_2 \cos(1,2) + \lambda_3 \cos(1,2) \cos(2,3) + \dots]$$

Since successive terms when averaged over scattering angles, will have increasing powers of $2/(3A)$, they will diminish rapidly in the approximation being considered, so that the various λ 's occurring in

successive terms may be replaced by λ_1 itself. This is valid if the λ 's change very little between successive collisions. In this manner, the coefficient of λ_1 is approximated by:

$$2[\lambda_1 + \lambda_1 C + \lambda_1 C^2 + \dots] = \frac{2\lambda_1}{1-C}; \quad C = \frac{2}{3A}$$

Consistent with our approximation, the finite series has been considered to be infinite. It can be readily shown that the error becomes negligible for $A \gg 1$ (and absorption processes are very few compared to scattering). Substitution of this approximation into (6-13) yields

$$(r^2)_{av} = \frac{2\lambda_1^2}{1-C} + \frac{2\lambda_2^2}{1-C} + \frac{2\lambda_3^2}{1-C} + \dots + \frac{2\lambda_n^2}{1-C} = \frac{2}{1-C} \sum_{i=1}^n \lambda_i^2$$

where $C = 2/(3A)$ as before. The sum may be written as an integral if the change of energy is small for each collision:

$$\begin{aligned} \text{Number of collisions between } E + \Delta E \text{ and } E &= \frac{\log_e(E + \Delta E) - \log_e E}{\xi} \\ &= \Delta E / (\xi E) \end{aligned}$$

Thus

$$(r^2)_{av} = \frac{2}{3A} \sum_{i=1}^n \lambda_i^2 = \frac{2}{\xi(1-\frac{2}{3A})} \int_E^{E_0} \frac{\lambda^2(E') dE'}{E'} \quad (6-14)$$

The integration variable is primed to distinguish it from the limits.

In (6-14) ξ is the average decrease in the natural logarithm of the energy per collision, defined in equation (6-7). The dependence of the mean free path on the energy has been noted by writing $\lambda^2(E)$. E_0 is the initial neutron energy at the source, and E is the energy of the neutron at the particular position r . Equation (6-14) the average displacement-squared for neutrons of energy E when the scattering is done by heavy nuclei.

It is often convenient to use logarithmic variables, in which case the average of the displacement-squared is:

$$(r^2)_{av} = \frac{2}{\xi(1-\frac{2}{3A})} \frac{\log_e E_0}{\log_e E} \int \lambda^2(\epsilon') d\epsilon' \quad \text{with } \epsilon' = \log_e E' \quad (A \gg 1) \quad (6-15)$$

As was mentioned previously, the formula for $(r^2)_{av}$ can be derived exactly. The result [Ricerca Sci. 7:13(1936)] is:

$$\begin{aligned} (r^2)_{av} &= 2\lambda^2(0) + 2\lambda^2(a) + 2 \int_0^a \lambda^2(x) dx + 2\lambda(0) \int_0^a \lambda(x) e^{-x/2} dx \\ &\quad + 2\lambda(0)\lambda(a)e^{-a/2} + 2\lambda(a) \int_0^a \lambda(x) e^{-(a-x)/2} dx \\ &\quad + 2 \int_0^a \lambda(u) du \int_0^{a-u} \lambda(u+x) e^{-x/2} dx \end{aligned} \quad (6-16)$$

where $x = \log_e(E_0/E')$; $a = \log_e(E_0/E)$

As in equations (6-15) and (6-14), the energy of the neutrons at the source is E_0 and the formula predicts the average of the displacement-squared for neutrons of energy E . E' is the variable of integration and disappears upon substitution of the limits.

Referring to Figure 18, page 37, we note that the cross section for neutrons in hydrogen is very nearly constant (~ 21 barns) over a wide range of energies - from about 1 ev (where the effect of chemical binding forces becomes negligible) to about 10 kev. For slowing down in this region, λ is constant, and the formula (6-16) can be reduced to the form:

$$r_{av}^2 = f(E_0) - 6\lambda^2 \log_e E \quad (6-17)$$

Using Ra-Be as a neutron source in a tank of water, we find the following data experimentally:

$$\text{Slowing to Rh resonance (1.28 ev): } r_{av}^2 = 276.6 \text{ cm}^2$$

$$\text{Slowing to I resonance (37 ev): } r_{av}^2 = 262.2 \text{ cm}^2$$

It follows then from equation (6-17), which is valid in this range of energies, that:

$$[(r^2)_{av} \text{ for Rh}] - [(r^2)_{av} \text{ for I}] = 6\lambda^2 \log_e (37/1.28) = 14.4 \text{ cm}^2$$

from which $\lambda^2 = (14.4/20.2 \text{ cm}^2)$ or $\lambda = 0.84 \text{ cm}$. This is an average mean free path, to be compared with appropriate averages of differential data. Recent data on Ra-Be neutrons in water [Phys. Rev. 73:271 (1948)] indicates that r_{av}^2 for indium (1.44 ev) is 272 cm^2 .

6.4 DISTRIBUTION OF NEUTRONS FROM POINT SOURCE - AGE EQUATION

In the last section, we have discussed one description of the space distribution of neutrons for a point source in an infinite homogeneous medium. In this section, we shall derive an expression for the neutron "age" rather than the average of the displacement-squared. This "age" is likewise a distance-squared. (The somewhat misleading nomenclature results from the analogy between the present problem and heat flow.)

As before, neutrons with energy E_0 are fed into the scattering medium. Our interest is in the space distribution of neutrons of various energies. Accordingly we define a steady-state neutron density function $n(x, y, z, \epsilon)$ such that $n(x, y, z, \epsilon) dx dy dz d\epsilon$ is the number of neutrons in the volume element $dx dy dz$, with the logarithm of their energy between ϵ and $\epsilon + d\epsilon$ (where, as before, $\epsilon = \log_e E$). Consider the volume element and the neutrons in it in the given energy range. Per unit time, this volume element will receive neutrons in this energy range from two sources: (1) from diffusion of neutrons of this energy from outside the volume element and (2) from higher energy neutrons in the volume element which have their energy degraded into the given energy range.

From the first source, diffusion, the contribution to the neutron population in the volume element can be calculated using methods of kinetic theory. The diffusion coefficient is:

$$D = \lambda_v v / 3 = \frac{\lambda v}{3 [1 - \cos \phi_{av}]} \quad (6-18)$$

where λ is the mean free path for scattering, $(\cos \phi)_{av}$ the average of cosines of the angle of scattering in the laboratory system, and v the neutron velocity. λ_v is called the "transport mean free path." It is the distance that a neutron would travel on the average in the direction of its initial motion after an infinite number of collisions, each collision resulting in an average deflection given by $(\cos \phi)_{av}$.

and each collision being followed by one mean free path (λ) movement. As shown in equation (6-18), λ_t and λ are related by the equation:

$$\lambda_t = \frac{\lambda}{1 - (\cos \phi)_{av}} \tag{6-19}$$

For isotropic scattering, equation (6-4), since $(\cos \phi)_{av} = 2/(3A)$, the relation between the transport mean free path and the mean free path is:

$$\lambda_t = \frac{\lambda}{1 - \frac{2}{3A}}$$

When $A \gg 1$, $\lambda_t \approx (1 + \frac{2}{3A}) \lambda \approx \lambda$. As in kinetic theory, the neutron current density S is related to the diffusion coefficient by:

$$S = -D \nabla n \text{ (neutrons/cm}^2\text{/sec)} \tag{6-20}$$

This is a vector equation, with S in the direction of the gradient. Notation is standard:

$\nabla n = i_x(\partial n/\partial x) + i_y(\partial n/\partial y) + i_z(\partial n/\partial z)$ with i_x , i_y , and i_z the unit vectors in the x , y , and z directions. Consider the face $dydz$ of the volume element at (x,y,z) . The neutron current out of this face (in the negative x -direction) is $D(\partial n/\partial x)$ so that the number of neutrons per unit time in the energy range between ϵ and $\epsilon + d\epsilon$ going out of the face $dydz$ at (x,y,z) is $D(\partial n/\partial x) dydz d\epsilon$. At the opposite face at $(x + dx, y, z)$ the number of neutrons per unit time in the energy range $d\epsilon$ coming into the volume element (negative x -direction) is:

$$D \left(\frac{\partial n}{\partial x} + \frac{\partial^2 n}{\partial x^2} dx \right) dydz d\epsilon$$

The net gain of neutrons per unit time in the volume element is thus $D(\partial^2 n/\partial x^2) dx dydz d\epsilon$. Taking other pairs of faces and adding up, we obtain:

$$\begin{aligned} \text{Neutrons in energy range between} \\ \epsilon \text{ and } \epsilon + d\epsilon \text{ diffusing into the} \\ \text{volume } dx dy dz \text{ per unit time} \end{aligned} = D \nabla^2 n dx dy dz \tag{6-21}$$

with ∇^2 the Laplacian $= (\partial^2/\partial x^2) + (\partial^2/\partial y^2) + (\partial^2/\partial z^2)$.

Now consider the other source of neutrons, those in the volume element which are slowed down to the proper energy range (between ϵ and $\epsilon + d\epsilon$). The number of collisions of a neutron in unit time is v/λ . If this is multiplied by the average change in ϵ per collision, that is by ξ , the result $\xi v/\lambda$ is the loss of ϵ per unit time. Representing values of ϵ as points on a straight line, Figure 38, a neutron can be visualized as moving down the ϵ line with a velocity $\xi v/\lambda$

$(\epsilon = \text{LOG}_e E)$

NEUTRON PROCEEDS DOWN ϵ AXIS WITH SPEED = $\xi v/\lambda$

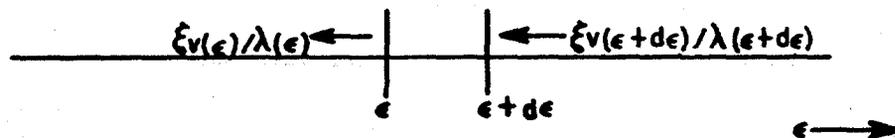


Figure 38. Slowing down of neutrons: appearance on ϵ axis.

A-93-6

(velocity meaning loss of ϵ per unit time). At the point ϵ , the number of neutrons in the volume element $dx dy dz$ which move out of the range of energies (between ϵ and $\epsilon + d\epsilon$) per unit time is:

$$[\xi v(\epsilon)/\lambda(\epsilon)] \cdot n(x, y, z, \epsilon) dx dy dz$$

where the functional dependent of v and λ on ϵ has been indicated. The number per unit time moving into the top of the energy range at $\epsilon + d\epsilon$ is:

$$\left[\xi v(\epsilon + d\epsilon)/\lambda(\epsilon + d\epsilon) \cdot n(x, y, z, \epsilon + d\epsilon) dx dy dz \right. \\ \left. \left\{ \xi \frac{v(\epsilon)n}{\lambda(\epsilon)} + \frac{\partial}{\partial \epsilon} \left[\xi \frac{v(\epsilon)n}{\lambda(\epsilon)} \right] d\epsilon \right\} dx dy dz \right]$$

The net increase per unit time of neutrons in the range $d\epsilon$ and in the volume element $dx dy dz$ is obtained by subtracting outgoing from incoming:

$$\text{Net increase} = \frac{\partial}{\partial \epsilon} \left[\xi \frac{v(\epsilon)n}{\lambda(\epsilon)} \right] \cdot dx dy dz d\epsilon \quad (6-22)$$

Combining equation (6-22) with equation (6-21) gives the slowing down differential equation. Since we are concerned with the steady-state (time-independent) neutron density, the sum of equations (6-21) and (6-22) must be zero:

$$D \nabla^2 n + \frac{\partial}{\partial \epsilon} K \left(\xi \frac{vn}{\lambda} \right) = 0 \quad (6-23)$$

$$\text{with } D = \frac{\lambda v}{3 [1 - (\cos \phi)_{av}]} = \lambda v/3$$

$$\lambda = \lambda(\epsilon), v = v(\epsilon), n = n(x, y, z, \epsilon), \epsilon = \log_e E$$

Equation (6-23) can be transformed into the same form as the classical heat conduction equation by the introduction of new dependent and independent variables. The dependent variable is the so-called "slowing down density" defined as:

$$q = \xi vn/\lambda \quad (6-24)$$

The name is descriptive. As pointed out earlier, the loss of ϵ per unit time per neutron is $\xi v/\lambda$ (Figure 38). Multiplied by the neutron density n , this is then the total loss in ϵ per unit time per unit volume per unit energy interval or, expressed differently, the number of neutrons per unit volume per unit time crossing any value ϵ on the ϵ axis. In the steady state, if q is integrated over all space, the number of neutrons crossing any value ϵ per unit time is certainly the number of neutrons fed into the system per unit time, i.e., a constant. Thus $\int q(x, y, z, \epsilon) dx dy dz = \text{constant}$ for all ϵ . If q is substituted into equation (6-23), the result is:

$$\left\{ \frac{\lambda^2(\epsilon)}{3\xi [1 - (\cos \phi)_{av}]} \right\} \nabla^2 q + \frac{\partial q}{\partial \epsilon} = 0 \quad (6-25)$$

This is a simpler form than equation (6-23) since the differential operators only operate on q . The independent variable can be changed to further simplify the form. Let us introduce the independent variable τ , the "age" (often called the "Fermi age"):

$$\tau = \frac{1}{3\xi(1 - \cos \phi)_{av}} \int_0^{\epsilon} \lambda^2(\epsilon') d\epsilon' \quad (6-26)$$

It is apparent that τ has the dimensions of length². Differentiating q with respect to ϵ :

$$\frac{\partial q}{\partial \epsilon} = \frac{\partial q}{\partial \tau} \frac{\partial \tau}{\partial \epsilon} = - \frac{\partial q}{\partial \tau} \left[\frac{\lambda^2(\epsilon)}{3\xi(1 - \cos \phi)_{av}} \right]$$

Substitute into equation (6-25) and cancel out the common factor. The result is the "Age Equation":

$$\nabla^2 q = \frac{\partial q}{\partial \tau} \quad (6-27)$$

with q and τ defined by equations (6-24) and (6-26).

A considerable advantage is secured by these transformations in that equation (6-27) is identical in form to the heat conduction equation:

$$\nabla^2 T = (c\rho/k) \frac{\partial T}{\partial t}$$

Thus q is analogous to temperature and τ to time. Just as temperature will decrease with increasing time (no heat sources), so will the neutron "born" with energy ϵ_0 lose its energy ϵ with increasing "age" τ . This can be seen by inspection of equation (6-26), where $\tau \rightarrow 0$ as $\epsilon \rightarrow \epsilon_0$ and τ increases with decrease in ϵ (or τ increases as the "time since birth" increases).

Let us apply the age equation to the point source problem. Given a point source of neutrons with energy E_0 in an infinite medium, what is the density of the neutrons as a function of E and position? First solve the age equation for q , borrowing the corresponding solution of the heat equation:

$$q = \frac{Q}{(4\pi\tau)^{3/2}} e^{-r^2/4\tau} \quad (6-28)$$

Identification of Q as the source strength, that is, the number of neutrons with energy E_0 introduced into the system in unit time, follows by integrating q over all space using spherical polar coordinates:

$$\int_0^{\infty} q \cdot 4\pi r^2 dr = 4\pi [Q/(4\pi\tau)^{3/2}] \int_0^{\infty} e^{-r^2/4\tau} r^2 dr = Q$$

since

$$\int_0^{\infty} \exp(-\alpha x^2) x^2 dx = \sqrt{\pi}/(4\alpha^{3/2}).$$

Let us examine the form of the solutions equation (6-28). In Figure 39, a sketch of q versus r for large and small values of τ shows that as E decreases from E_0 (i.e., as τ increases), the space

* Although ϵ is conveniently referred to as the neutron's energy, it should be remembered that ϵ is the natural logarithm of the neutron energy.

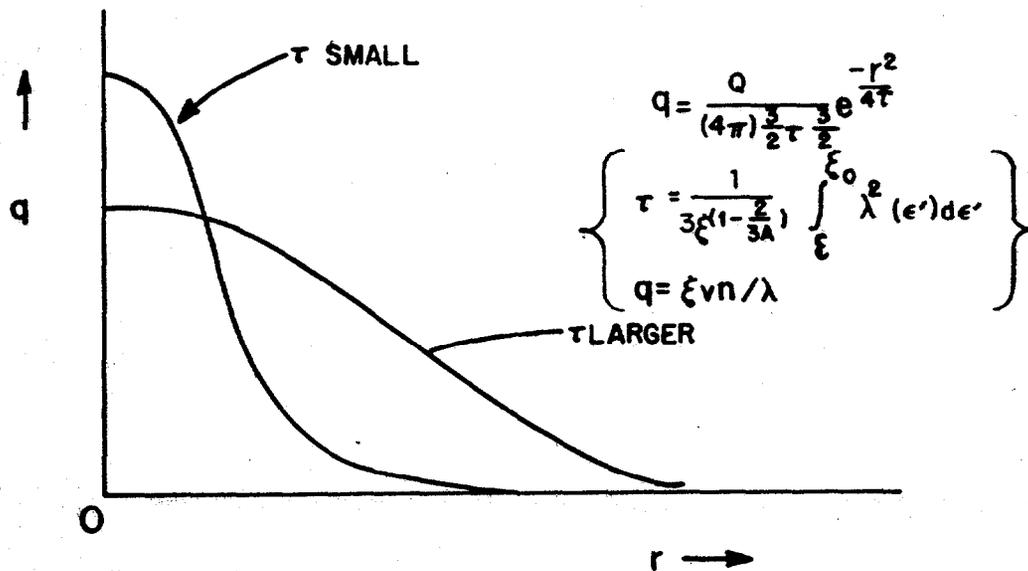


Figure 39. Point Source Solution of age equation.

distribution for the energy E gets broader and broader. This is as one would expect. Fast neutrons are distributed close to the source, and slow neutrons are spread out. (Note that the area under the curves between r and $r + dr$ in the figure does not give the number of neutrons per unit time arriving in that interval. A factor $4\pi r^2$ must be introduced. In fact, $4\pi r^2 q dr$ is the number of neutrons per unit time arriving in the space interval between r and $r + dr$.)

In problem 5 at the end of this chapter, the average of r^2 is calculated from q . The result, 6τ , is similar to that obtained in Section 6.3.

PROBLEMS

1. Consider the collision of neutrons with beryllium. What is the average angle of scattering observed in the laboratory frame of reference? In the center of gravity system?
2. Calculate ξ for H^2 , He^4 , Be^9 , O^{16} , U^{238} . In each case, how many collisions will be needed to reduce a neutron's energy from 1 Mev to 1 ev?

3. Suppose a point source of fast neutrons is placed in a large tank of water. At various distances from the source, indium foils (sandwiched between cadmium foils) are exposed to the neutron flux and thereby activated. Exposure times and foil areas are constant. The following data are obtained (after correction for activity decay, etc.):

r(cm)	6	8	10	12	14	16	18	20	22	24	26	28	30
A(In)	890	550	302	180	101	61	40	26	18	11.5	8.7	5.7	4.0

r is the distance from the source to the indium detector, and A is the activity in counts per minute of the indium detector (activity is due to 1.44 eV neutrons). From these data, calculate $(r^2)_{av}$. (N.B. The activity A is not proportional to the number of neutrons in the interval between r and $r + dr$.) Values of A for larger r can be determined by semilogarithmic extrapolation.

4. Consider a substance in which λ is constant. In this case, what is the relation between the age τ and the actual "time from birth"?

5. Using the point source solution for q , find $(r^2)_{av}$ in terms of τ . Apply this result to the hydrogen problem discussed in the sections before and after equation (6-17). Show that the same values are obtained using age theory as in the vector averaging process.

CHAPTER VII

THE DISTRIBUTION OF SLOW NEUTRONS IN A MEDIUM

7.1 THE DIFFERENTIAL EQUATION FOR SLOW NEUTRONS

Calculation of the distribution of neutrons of various energies in a medium involves two distinct problems. First, there is the slowing down problem dealt with in the last chapter. The neutrons do not continue to be slowed down indefinitely, for the nuclei they collide with are not at rest but have vibrational energies corresponding to their temperature. Eventually the neutrons come into thermal equilibrium with these nuclei and show a Maxwellian distribution* corresponding to the temperature of the medium. Clearly the problem of the spatial distribution of these slowed down or thermal neutrons is quite distinct from that of the distributions of neutrons being slowed down and must be handled by different methods.

In approaching this second problem, we ask, "Given a source of thermal neutrons, what can be said about their stationary state distribution in a medium?" We seek a differential equation as our description. Let $n(x,y,z)$ be the density of thermal neutrons at x,y,z . As in the derivation of the age equation, we consider a unit volume. There are three mechanisms by which the number of neutrons in this volume element change with time: (1) diffusion into or out of the volume element, (2) absorption † or capture of neutrons in the volume element, and (3) generation of thermal neutrons in the volume element by the slowing down of fast neutrons to thermal energies. The first mechanism will yield $D\nabla^2 n$ neutrons per unit time per unit volume, as in deriving equation (6-21) of Chapter VI. The second mechanism decreases the neutron density per unit time by $-n/\theta$, where θ is the mean time for absorption or capture. The third is just the slowing down density q evaluated for thermal energies, since $q(\epsilon)$ is the number of neutrons per unit volume per unit time arriving at a particular logarithmic energy ϵ . To emphasize that q is to be evaluated for thermal energies, we write q_T . Adding the three contributions together gives the differential equation for the time rate of change of the neutron density:

$$D\nabla^2 n - (n/\theta) + q_T = \partial n/\partial t \tag{7-1}$$

where $n(x,y,z,t)dx dy dz$ is the number of thermal neutrons in the volume element $dx dy dz$ at time t , $(n/\theta)dx dy dz$ is the number of thermal neutrons absorbed per second in the volume element $dx dy dz$ at time t , and $q_T dx dy dz$ is the number of thermal neutrons created (by slowing down) per unit time in $dx dy dz$. It should be remembered that q_T is a function of x,y,z which can be determined from the age equation with the proper boundary conditions. Since $D = \lambda_e v/3$, equation (7-1) can be rewritten for the steady state:

$$\nabla^2 n - \frac{3}{\lambda_e v \theta} n + \frac{3q_T}{\lambda_e v} = 0$$

By defining:

$$A = v\theta = \text{"capture mean free path"}$$

$$L = \sqrt{\Lambda \lambda_e/3} \text{ "diffusion length"} \tag{7-2}$$

*Number of neutrons with velocities between v and $v + dv$ is proportional to $v^2 \exp(-mv^2/2kT) dv$.
 †This mechanism should perhaps also have been considered in the slowing down process. But whereas orders of magnitude are such that in the slowing down the consideration of absorption is usually a refinement, here it is a necessity.

A-93-7.

the steady state equation becomes:

$$\nabla^2 n - \frac{n}{L^2} + \frac{3q_T}{\lambda_v} = 0 \quad (7-3)$$

For a point source of slow neutrons ($q_T = 0$ except at $r = 0$ where q_T is a delta function), the equation reduces to:

$$\nabla^2 n - (n/L^2) = 0 \quad (7-4)$$

The solution is spherically symmetrical, so equation (7-4) reduces to the radial equation:

$$\frac{1}{r} \frac{d}{dr} \left(r^2 \frac{dn}{dr} \right) - \frac{n}{L^2} = 0 \quad (7-5)$$

Let $u = nr$. The equation becomes $u'' - (u/L^2) = 0$ where $u'' = d^2u/dr^2$. Solutions are $u = \exp(\pm r/L)$. The boundary condition that $n \rightarrow 0$ as $r \rightarrow \infty$ eliminates the positive exponential, so the solution is:

$$n = A \frac{e^{-r/L}}{r}$$

The constant A can be evaluated by considering a small sphere enclosing the source at the origin. The neutron current, equation (6-20) of Chapter VI, is $-D\nabla n = -Ddn/dr = -DAe^{-r/L} [-1/(Lr) - 1/r^2]$. Multiplying this by the area of the sphere $4\pi r^2$ and letting $r \rightarrow 0$ gives $4\pi DA$. This is the source strength Q . Thus A is $Q/(4\pi D)$ or $3Q/(4\pi\lambda_v)$. The complete solution to (7-3) for a point source of Q slow neutrons per unit time is:

$$n = \frac{3Q}{4\pi\lambda_v} \left(\frac{e^{-r/L}}{r} \right) \quad (7-6)$$

The solution may be checked by substitution in equation (7-5).

The point source solution is particularly important since any source can be represented by a proper assembly of point sources, and the corresponding solution is the superposition of these point source solutions.

To solve this equation for a *point source of fast neutrons* we set up an integral over a distribution of thermal neutron point sources all over space that arise from the slowing down of the fast neutrons (see Figure 40). Now we can find the density of slow neutrons at a distance r from the point source of fast neutrons as follows. In any volume element dV , a distance $\bar{\rho}$ from the point source of fast neutrons, there are $q_T dV$ thermal neutrons per second being produced by the slowing down process where q_T [see equation (6-28)] is:

$$q_T = \frac{Q}{(4\pi\tau)^{3/2}} e^{-\rho^2/4\tau}$$

with τ = age for thermal neutrons,

Q = fast neutron source strength (neutrons/second).

The volume element dV is expressed in spherical coordinates, $dV = \rho^2 d\rho \sin \theta d\theta d\phi$, and the polar axis ($\theta = 0$) is taken to pass through the point at which we are finding the thermal neutron

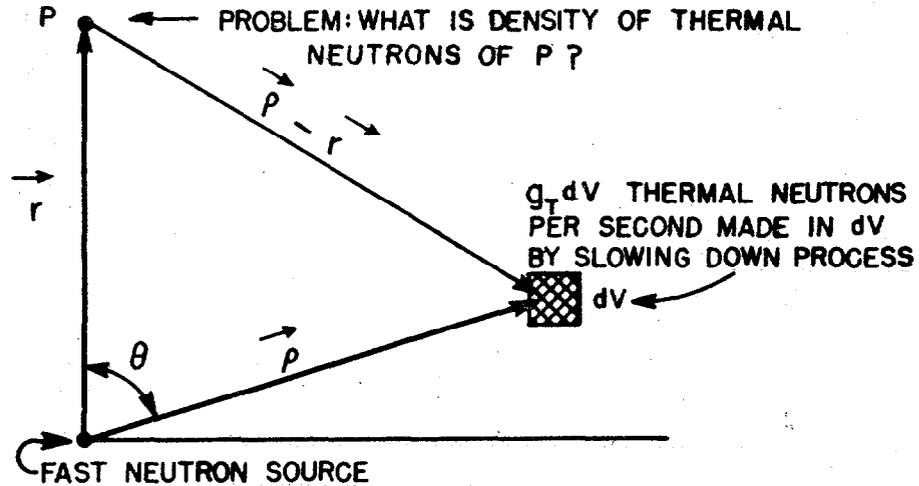


Figure 40. Point source of fast neutrons.

density (see point P in Figure 40). Now point P is a distance $|\vec{\rho} - \vec{r}| = \sqrt{\rho^2 + r^2 - 2\rho r \cos \theta}$ from the source $q_T dV$. As a consequence, the density of thermal neutrons observed at P from this source is given by equation (7-6) with the appropriate source strength and radial distance substituted:

$$\left(\begin{array}{l} \text{Thermal neutrons at P} \\ \text{from source at dV} \end{array} \right) = \left(\frac{3q_T dV}{4\pi\lambda_g v} \right) \cdot \frac{e^{-|\vec{\rho} - \vec{r}|/L}}{|\vec{\rho} - \vec{r}|}$$

Adding up the contributions from all sources means we integrate over dV . Substituting for q_T , $|\vec{\rho} - \vec{r}|$, carrying out the integration over the azimuthal angle ϕ and changing variable $\mu = \cos \theta$ finally gives:

(Thermal neutron density at distance r from fast neutron source of strength Q)

$$n(r) = \frac{3Q}{2\lambda_g v (4\pi\tau)^{3/2}} \int_{\rho=0}^{\infty} \int_{\mu=-1}^{+1} e^{-\rho^2/4\tau} \frac{e^{-\sqrt{\rho^2 + r^2 - 2\rho r \mu}/L}}{\sqrt{\rho^2 + r^2 - 2\rho r \mu}} \rho^2 d\rho d\mu \quad (7-7)$$

In equation (7-7) $\lambda_g v$, τ , and L are the transport mean free path, velocity, age, and diffusion length for thermal neutrons.

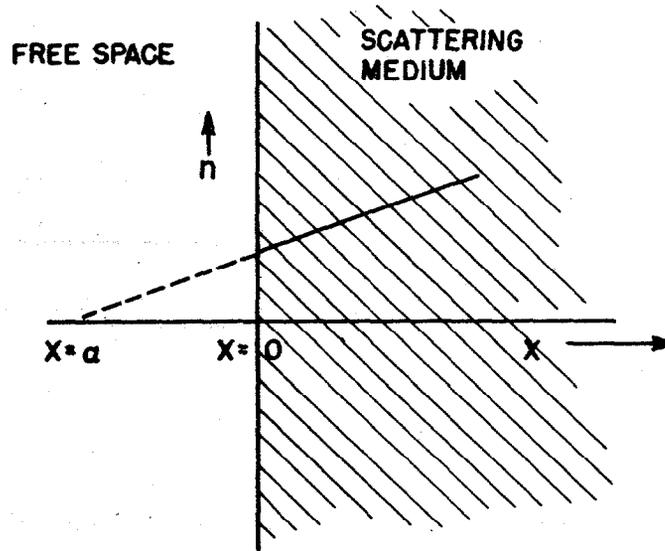


Figure 41. Neutron density behavior at boundary surface.

7.2 BOUNDARY CONDITIONS FOR THE SLOW NEUTRON DIFFERENTIAL EQUATION

In order to solve the slow neutron differential equation, the behavior of n or some function of n at the spatial boundaries must be known. Consider a finite convex (i.e., not re-entrant) medium with a neutron source in it and free space everywhere around it. What can be said of q or n at the bounding surface? To a first approximation, q or n can be taken equal to zero. This is made somewhat plausible by the argument that free space acts as a perfect sink; namely, it absorbs all neutrons and returns none. It, therefore, acts as such a heavy drain on the neutron density at the boundary that no density can be maintained there.

Actually it can be shown that a more proper boundary condition is that n or q vanish at a surface $(2/3)\lambda_t$ away from the bounding surface,* where λ_t is the neutron transport mean free path in the medium. Consider a plane bounding surface, Figure 41. The neutron density in the neighborhood of the boundary can be approximated by a linear function of the distance: $n = p(a + x)$. This can be shown to satisfy equation (7-4) for a one-dimensional situation. The flux at the bounding surface is just $D\nabla n$ [see equation (6-20)] in the negative x direction. Since the gradient of n is just dn/dx , the flux is $D(dn/dx) = D \cdot p = (\lambda_t v/3)p$. The flux can be calculated by another means. In Figure 42, it is apparent that the probability that a neutron coming from the unit volume ΔV will reach the surface at point P is $\exp(-x/\cos \theta / \lambda_t)$. Moreover, the fraction of the total solid angle included between θ and $\theta + d\theta$ is $\sin \theta d\theta/2$ [see equation following equation (6-5)]. Since the volume ΔV is the source of $n\Delta V/(\lambda_t v)$ neutrons per unit time (an equal number of neutrons return to the volume per unit time in the steady state), then the total number of neutrons crossing the boundary per unit time and coming from the volume ΔV is:

$$\int_{\theta=0}^{\pi/2} \exp\left(-\frac{x/\cos \theta}{\lambda_t}\right) (\sin \theta d\theta/2) (nv\Delta V/\lambda_t)$$

*A more refined derivation gives $0.71 \lambda_t$.

a-93-7.

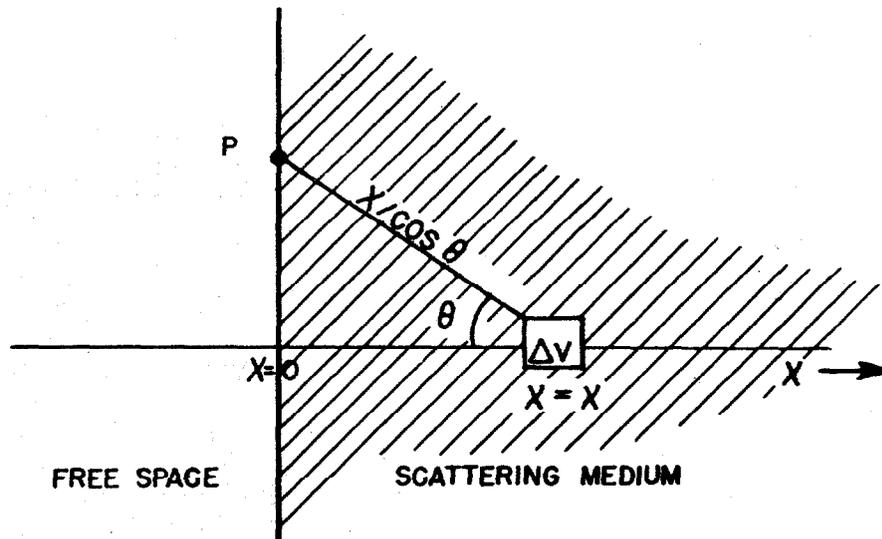


Figure 42. Neutron flux calculation at boundary.

Substituting for $n [= p(\alpha + x)]$ and rearranging this equation becomes:

$$(1/2) \int_{\theta=0}^{\pi/2} p(\alpha + x)(v/\lambda_t) e^{-x/(\lambda_t \cos \theta)} \sin \theta \, d\theta \, \Delta V$$

Consider the volume to have unit area perpendicular to the x axis and depth dx . Then integrating the foregoing expression over x would give the total number of neutrons each second coming to the surface from an infinite column of unit cross-sectional area perpendicular to the surface. But this is seen to be the flux or the number of neutrons crossing unit area of the surface per second, provided, of course, that everything can be assumed uniform perpendicular to the x direction. Thus the flux is:

$$(1/2) \int_0^{\infty} \int_0^{\pi/2} p(\alpha + x)(v/\lambda_t) e^{-x/(\lambda_t \cos \theta)} \sin \theta \, d\theta \, dx$$

Change variables to $\mu = \cos \theta$ and integrate first over x , then over μ . The result is $(vp/2\lambda_t) [(a\lambda_t/2) + (\lambda_t/3)]$. Equating this to the previous result for the flux, $p\lambda_t v/3$ and solving for a gives $a = (2/3)\lambda_t$. But a is the x intercept of the neutron flux. Therefore, we have shown that n vanishes at a distance $(2/3)\lambda_t$ outside the bounding surface.

It must be noted that this boundary condition really describes n at the boundary, not beyond the boundary. In particular, the boundary condition does not mean that n vanishes at $x = -(2/3)\lambda_t$ and is negative beyond that point. What has been proved is simply that the density at a bounding surface behaves as though n is a linear function of x , vanishing at $x = -(2/3)\lambda_t$.

7.3 THE DIFFUSION LENGTH IN WATER AND GRAPHITE

To be able to use the slow neutron diffusion equation (7-3), it is necessary to know the value of L , the diffusion length. Measurement of L for water can be accomplished by placing a water tank on a pile (or on the thermal column of a pile), as in Figure 43. Provided diameter and height of water are very much greater than the diffusion length itself, the problem may be considered to be one dimensional. The bottom surface is a plane source of thermal neutrons. Equation (7-3) becomes:

$$\frac{d^2n}{dx^2} - \frac{n}{L^2} = 0$$

$$n = n_0 e^{-x/L}$$
(7-8)

where x is the distance from the bottom of the water tank. Measurements are made with and without cadmium separating the pile and water tank to determine (by subtraction) n for thermal neutrons as a function of x . The exponential decrease of n with x as predicted in equation (7-8) is measured and L determined for water. The value is $L_{\text{water}} = 2.8 \text{ cm}$. If a block of paraffin is used, the result is the same, showing that the absorbing mechanism in the case of water and paraffin is the same; namely, hydrogen capture. The effect of carbon or oxygen absorption is negligible compared to hydrogen.

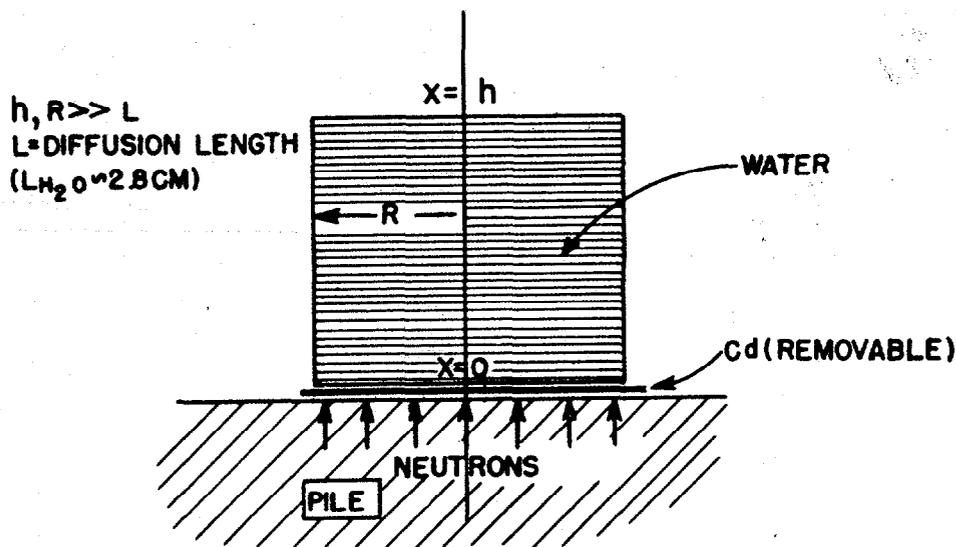


Figure 43. Measurement of diffusion length in water.

A-93-

There are other ways to measure L and related constants for water. Some are described in a paper by Fermi and Amaldi, Phys. Rev. 49:899 (1936). The diffusion length is dependent upon temperature. For water, the relation is:

$$L = 2.64 + 0.0061 T$$

(L in cm, T in °C) (7-9)

As shown in equation (7-2), the diffusion length is related to the mean free path for absorption (Λ) and the transport mean free path (λ_t). Knowing L and λ_t , we can calculate Λ , or vice versa.

For a substance such as graphite, the diffusion length is so large as to make the method just described impractical for determination of L . The one-dimensional approximation will not be valid when L is of the order of the dimensions of the medium. As a consequence, the three-dimensional problem must be solved.

The physical arrangement is shown in Figure 44. A fast neutron source is at point P ($u, u, 0$) on the bottom surface of the graphite pile. The height of the pile is much larger than the diffusion length, whereas the edge dimensions "a" are of the same order of magnitude as L . The slowing down equation, $\nabla^2 q - (\partial q / \partial \tau) = 0$, can be solved for this arrangement by use of Fourier analysis (same methods as used in solution of heat conduction problems). The result is:

$$q = (4/a^2)(Q/\sqrt{4\pi\tau}) e^{-z^2/4\tau} \sum_{r,s=1}^{\infty} e^{-\pi^2\tau(r^2+s^2)/a^2} \sin(\pi r x/a) \sin(\pi s y/a)$$

(Q = source strength = fast neutrons/sec) (7-10)

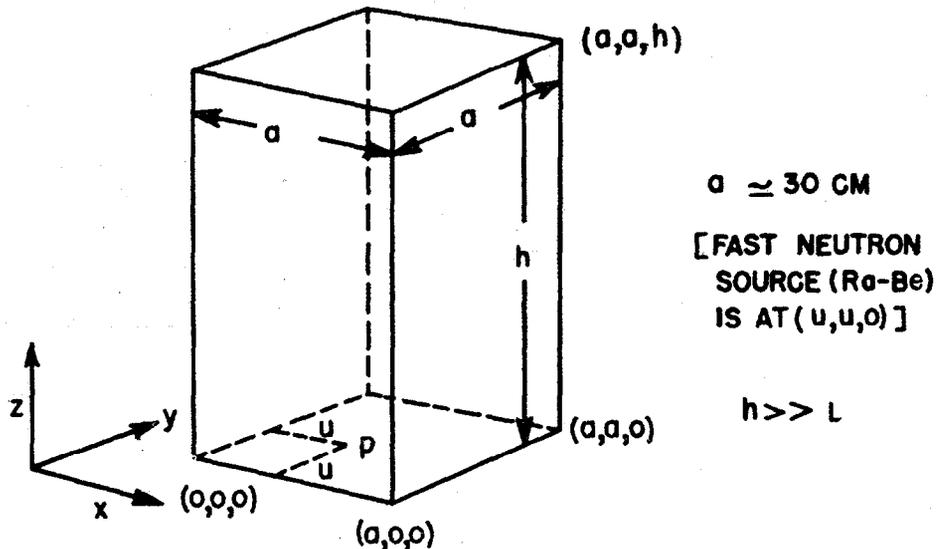


Figure 44. Measurement of diffusion length in graphite.

Evaluation of τ by experiment using a Ra-Be neutron source shows that the source has three well-defined neutron energies, giving three superposed distributions. The following table gives the age values for the three components in graphite at different energies and the percentage of each component present.

TABLE 2 - Ra-Be NEUTRONS IN GRAPHITE

PER CENT	τ (INDIUM RESONANCE), CM ²	τ (IODINE RESONANCE), CM ²
15.0	130	54
69.3	340	268
15.7	815	736

Such data are sometimes given in terms of the range r_0 , which is equal to $2\sqrt{\tau}$.

From the data of Table 2, it is apparent that at 30 cm or so from the source, q will be very small. In this case, the "source" term in the slow neutron diffusion differential equation (7-3) will be small. The slow neutron density n will be a solution to equation (7-4), i.e.,

$$\nabla^2 n - (1/L^2)n = 0$$

at such distances from the source. Because of the boundary conditions (the slow neutrons are produced by slowing down from the Ra-Be source), the solution is assumed to have the form:

$$n = \sum_{r,s=1}^{\infty} n_{rs}(z) \sin(\pi r x/a) \sin(\pi s y/a)$$

Substituting this into the differential equation gives an equation for $n_{rs}(z)$:

$$\frac{d^2 n_{rs}}{dz^2} - \left[\frac{\pi^2}{a^2} (r^2 + s^2) + \frac{1}{L^2} \right] n_{rs} = 0$$

This equation has a simple exponential solution, $\exp(-z/b_{rs})$, with b_{rs} equal to the reciprocal of the square root of the expression in brackets. Thus, the solution for n is:

$$n = \sum_{r,s=1}^{\infty} e^{-z/b_{rs}} \sin(\pi r x/a) \sin(\pi s y/a) \tag{7-11}$$

with $(1/b_{rs}^2) = (\pi^2/a^2)(r^2 + s^2) + (1/L^2)$

It can be seen that as r and s increase the exponential damps out rapidly with increasing z . Experimentally it suffices to compare activation measurements by slow neutrons in such a column for the $r = s = 1$ component:

$$n = e^{-z/b_{11}} \sin(\pi x/a) \sin(\pi y/a) \tag{7-12}$$

with $(1/b_{11}) = \sqrt{(2\pi^2/a^2) + (1/L^2)}$

For a typical graphite sample, the constants are: density = 1.551 gm/cm³; $b_{11} = 28.38$ cm; $a = 150.49$ cm. Since the neutron density does not vanish exactly at the edge, we must add

a-93-7;

$2 \times (2/3) \lambda_t$ to the 150.49 cm, making $a = 153.29$ cm ($\lambda_t = 2.1$ cm). As L depends on density, it is conventional to reduce all values of L to the value L would be if the density were 1.60 gm/cm³. This makes it possible to compare directly different lots of graphite tested in this way.

In the following table, some of the results obtained in this manner for four common substances are listed.

TABLE 3 - DIFFUSION LENGTHS AND RELATED CONSTANTS FOR H₂O, D₂O, Be, AND C

	DENSITY, G/CM ³	ATOMS/CM ³	L, CM	L ² /Λ, CM
H ₂ O	1.0	0.0334 × 10 ²⁴	2.85	0.142
D ₂ O	1.1	0.0331 × 10 ²⁴	100	0.80
BE	1.8	0.1235 × 10 ²⁴	31	0.87
C	1.62	0.0871 × 10 ²⁴	50.2	0.903

The microscopic constants in problems such as those just described are usually two: σ_s , the scattering cross section, and σ_a , the absorption cross section. These are related to L , θ , and λ_t in the following manner:

$$\lambda_t = \frac{\lambda}{1 - (\cos \phi)_{av}}, \quad \lambda = \frac{1}{n\sigma_s}$$

$$v\theta = \Lambda = \frac{1}{n\sigma_a}; \quad N = \frac{\sigma_s}{\sigma_a} = \frac{\Lambda}{\lambda} \quad (7-13)$$

$$L^2 = \frac{\lambda_t \Lambda}{3}$$

Definitions of the different symbols are:

- n = atoms/cm³
- v = neutron velocity
- θ = mean lifetime for absorption
- $(\cos \phi)_{av}$ = average of the cosine of the angle of scattering in the lab system = $2/(3A)$ for isotropic scattering (A is mass number)
- λ_t = transport mean free path
- λ = mean free path for scattering
- = absorption mean free path
- L = diffusion length
- N = average number of scattering collisions made per absorption

7.4 THE ALBEDO OR THE REFLECTIVITY OF BOUNDING SURFACES FOR NEUTRONS

So far in this chapter, we have outlined the methods of obtaining neutron distributions in media due to sources within them. However, neutrons are often introduced into a medium from the outside. It is convenient to define a reflectivity, or, as it is called, albedo ("whiteness" in Latin), for a surface. It is simply the fraction of the incident neutrons eventually returned or "reflected" from the surface. An albedo of unity means perfect reflection; an albedo of zero

means perfect (black body) absorption.*

Let us solve a typical problem. We shall calculate the albedo of an infinite plane surface for slow neutrons. But first it will be necessary to solve the following problem: "Given a medium bounded at $x = 0$ and occupying all of space to the right of this plane, what is the probability that a slow neutron starting at a point d units distant from $x = 0$ will escape from the medium, i.e., will reach $x = 0$?"

We shall do this problem twice, using two very different approaches. First, we shall use the slow neutron diffusion equation and assume that the problem is one dimensional, i.e., that the neutrons move only in the x direction. The second method will involve the solution of an integral equation.

Assume a point source of neutrons at $x = d$ on the x axis. We must calculate the flux at the origin for unit source strength. This is precisely the escape probability, since the boundary of the medium is at the origin. The slow neutron diffusion equation is in this case:

$$\nabla^2 n - \frac{n}{L^2} = 0$$

(everywhere but at $x = d$), and since this is a one-dimensional problem, the solution is:

$$n = Ae^{-x/L} - Ae^{+x/L}$$

for $0 \leq x < d$, assuming the boundary condition $n(x = 0) = 0$. For $x > d$, $n = Be^{-x/L}$. These two solutions must join at $x = d$ (see Figure 45) so that n is continuous and the gradient dn/dx

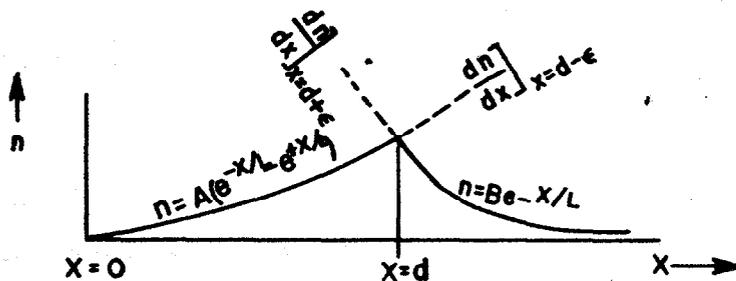


Figure 45. Joining solutions at discontinuity.

has a finite discontinuity. To find the amount of this discontinuity, integrate the complete diffusion equation (7-3) throughout a small "volume" surrounding the source:

$$\int_{d-\epsilon}^{d+\epsilon} \left[\frac{d^2 n}{dx^2} - \frac{n}{L^2} \right] dx = - \int_{d-\epsilon}^{d+\epsilon} \frac{q}{D} dx$$

The first term is $[dn/dx]_{d-\epsilon}^{d+\epsilon}$, the discontinuity in dn/dx . The second term vanishes as $\epsilon \rightarrow 0$, for n must, for physical reasons, be a continuous function of x . The last term is simply

*See Fermi's paper on the motion of neutrons in hydrogenous substances in *Ricerca Sci.* 7:13 (1936).

A-93-1

$-Q/D$ where Q is the source strength at $x = d$ and D is the diffusion coefficient $\lambda_c v/3$. Thus, the conditions on n and dn/dx at d are:

$$\text{(Continuity of } n) \quad Ae^{-d/L} - Ae^{+d/L} = Be^{-d/L}$$

$$\text{(Discontinuity of } -Q/D \text{ in } dn/dx) \quad -(A/L)e^{-d/L} - (A/L)e^{+d/L} = -(B/L)e^{-d/L} + (Q/D)$$

Hence $A = -LQe^{-d/L} / 2D$ and $B = LQ(1 - e^{-2d/L})$.

The flux at $x = 0$ is then $D (dn/dx)_{x=0} = Qe^{-d/L}$, and for unit source strength it is simply $e^{-d/L}$. This gives the probability, $p(d)$, that a slow neutron at a distance d from bounding surface will eventually escape from the medium.

Before we use this result to find the albedo of such a one-dimensional medium for neutrons incident on the boundary from the outside, let us do the problem in another way. In addition to the methods of diffusion, there is a more exact and rigorous way to attack problems of the type being discussed. To find the neutron density in a particular volume V at a time t , one could investigate the density of neutrons that are moving toward V and are in the other volumes at various earlier times t' so that (considering their velocities and distances from V) they would be in V at the time t . The neutron density at V at the time t could be expressed as some sort of sum or integral of these other neutron densities. We would be led to an integral equation in the neutron density n .

Thus in addition to the differential equation method of solving diffusion problems, there is an integral equation method, too. It would be well to stop a moment and compare the relative merits of the two approaches. In setting up the diffusion differential equation, it had to be assumed the quantities such as n , dn/dx , etc. vary slowly with respect to the mean free path of the diffusing particles. Further, it was assumed that densities of particles were large enough so that speaking of quantities such as dn/dx made sense. In particular, one would not expect that the solution of a problem like the following by diffusion methods would give physically true results: "Find $n(r, \theta, \phi)$, the density of slow neutrons in a sphere of radius $\lambda/2$, if there is a point slow neutron source at the center of the sphere and λ is the mean free path of slow neutrons in the medium of the sphere." There are, however, no such restrictions on the use of integral equation methods. No assumptions about the variation of dn/dx with distance, etc., need be made. Integral equation methods are more general and usually more difficult. It often becomes expedient to do diffusion problems by means of differential equations and proper boundary conditions first in order to get a rough idea about the function in question. Then the more exact solution can be obtained by means of an integral equation. This is the procedure we shall follow here. We have obtained $p(d)$, the probability that a neutron d units from the boundary of a one-dimensional medium will escape it, by means of the diffusion differential equation. Let us now apply integral methods.

Consider a neutron at d . As it leaves d , one of two things may occur. It may go to the left (toward the boundary), or it may go to the right, each with a 50% chance. If it goes to the left, it may escape before it suffers a collision, or it may experience a collision. The probability that a neutron at d will escape without a collision is, therefore, the product $(1/2) \times e^{-d/\lambda}$, where λ is the mean free path between collisions (mfp for scattering when $\sigma_s \gg \sigma_a$). However, a neutron may escape even if it suffers a collision. Suppose the neutron suffers its first collision

at x and that $\pi(x)$ is the probability that a neutron from d suffers its first collision at x (either to right or left of d). Now $p(x)$ is the probability that a neutron at x will eventually escape. The probability $p(d)$ that a neutron at d can escape is made up of two terms, the first of which is the probability of escape without collision, and the second is the sum of all possible products of $\pi(x)p(x)$, i.e., probability that the neutron is scattered to x times the probability of escape from x :

$$p(d) = \frac{1}{2}e^{-d/\lambda} + \sum_{\text{all } x} \pi(x)p(x)$$

It is assumed in the equation that there is isotropic scattering in the lab system, i.e., that $p(x)$ depends only on x and not on the side from which the neutron arrives at x . In detail, the sum should be written:

$$\frac{1}{2} \int_0^d e^{-(x-d)/\lambda} \frac{dx}{\lambda} p(x) \frac{N-1}{N} + \frac{1}{2} \int_0^\infty e^{-(x-d)/\lambda} \frac{dx}{\lambda} p(x) \frac{N-1}{N} \quad (7-14)$$

where the first integral gives the probability that a neutron starting from d will go left, suffer a collision at x , but will eventually escape, and the second integral gives the probability for the same thing with initial motion to the right. N in equation (7-14) is the ratio σ_s / σ_a given in equation (7-13), the average number of scattering collisions made per absorption. The probability that the collision at x is a scattering is $\sigma_s / (\sigma_s + \sigma_a)$ or $N / (N + 1) \approx (N - 1) / N$ for $N \gg 1$. Simplifying equation (7-14) by combining the integrals and adding the no-collision escape probability gives:

$$p(d) = \frac{1}{2} e^{-d/\lambda} + \frac{1}{2} \int_0^\infty e^{-|x-d|/\lambda} \frac{dx}{\lambda} p(x) \frac{N-1}{N} \quad (7-15)$$

Since the solution of the differential equation gave $p(d) = e^{-d/L}$, we therefore try $p(d) = Ae^{-\alpha d}$ as the solution of this integral equation. Doing this, we find that:

$$p(d) = \frac{\sqrt{N}}{\sqrt{N+1}} e^{-\frac{d}{\lambda \sqrt{N}}} \quad (7-16)$$

From equation (7-13) we note that $N = \Lambda / \lambda = 3L^2 / \lambda_t \lambda$. Expressing the exponential of equation (7-16) in terms of L gives $\exp(-d \sqrt{\lambda_t} / \lambda / \sqrt{3L})$, which is to be compared to the exponential $\exp(-d/L)$ derived by the differential equation method.

We are now ready to attack the original problem of finding the albedo. Say a beam of slow neutrons moving along the x axis from the left (negative x) hits the plane $x=0$. The probability that a neutron of the beam will make its first collision in dx at x is $e^{-x/\lambda} (dx/\lambda)$. The probability of not being absorbed and escaping from here is $p(x) \frac{N-1}{N}$. Hence the albedo is:

$$\beta = \int_0^\infty e^{-x/\lambda} \frac{dx}{\lambda} p(x) \frac{N-1}{N} = \frac{\sqrt{N-1}}{\sqrt{N+1}} \quad (7-17)$$

A nonabsorbing medium ($N \rightarrow \infty$) would eventually return all neutrons and have an albedo of 1.

If we wish to know β for an angle of incident θ we are forced to drop the one-dimensional attack and the problem becomes more difficult. The result is:

$$\beta(\theta) = \frac{\sqrt{N-1}}{\sqrt{N+3} \cos \theta} \quad (7-18)$$

This is inconsistent with the first result at $\theta = 0$, only because this solution allows for the fact that although $\theta = 0$ for the incident neutrons, they are not restricted to move along the x axis in the medium. The effect of allowing motion at angles to the x axis is to allow longer paths and hence more chance for absorption. This makes β slightly smaller.

Suppose we wish to measure the albedo for slow neutrons on paraffin. A direct measurement would be difficult, for even if a collimated slow neutron beam can be made to impinge on some paraffin, neutrons would be coming off at all angles from all over the surface of the paraffin, and their detection would not be easy. A much simpler way to find the albedo is the following. Place a thin-foil* slow neutron detector (Figure 46) somewhere in the middle of a mass of paraffin whose boundaries are far enough away from the foil that the paraffin can be considered infinite in extent. By means of some neutron source, we induce an activity in the foil. Call this activity A. Next back the foil on one side with some cadmium, enough that the cadmium will absorb practically all the slow neutrons hitting it but not enough to distort the neutron flux field appreciably. Measure the new activity in the detector foil. Call this activity B. Now the ratio (A/B) bears a simple relation to the albedo. To understand this, consider the number of neutrons ν hitting the foil each second in situation B. It is clear that for a uniform distribution of slow neutrons, the foil in case A would have ν neutrons per second hitting it from each side and would have at least 2ν slow neutrons hitting it per second. Actually, more than 2ν slow neutrons will hit the foil in case A, for some of the neutrons passing through the foil can return and pass through it again, there being no cadmium about to prevent this (see Figure 47). In fact, we can calculate the average number of times a neutron about to hit the foil will pass through it before it is eventually absorbed in the paraffin. Certainly the probability that this neutron will return through the foil is β , the albedo of the paraffin for slow neutrons. The probability that it will make at least two trips is $\beta \times \beta$ or β^2 , and so on. Thus the total number of passages through the foil for a neutron about to hit it is on the average:

$$1 + \beta + \beta^2 + \dots = \frac{1}{1 - \beta}$$

Hence, there would be $2\nu / (1 - \beta)$ slow neutrons hitting the foil each second, rather than simply 2ν neutrons. From this it follows that:

$$\frac{A}{B} = \frac{2\nu / (1 - \beta)}{\nu} = \frac{2}{1 - \beta}; \quad \beta = 1 - 2(B/A) \quad (7-19)$$

In this manner a measurement of the two activities A and B suffices to determine the albedo. For paraffin, A/B is 11. From equation (7-19), it follows that the albedo for paraffin is $\beta = 0.82$. This type of measurement would not be feasible with poor absorbers of neutrons, for it has to be assumed that the diffusion length in the medium is small compared to the foil size, i.e., that most of the "reflection" takes place close to the foil. Further, if the detector is not thin, it acts as its own cadmium, so to speak, and a correction must be made for the absorption in the foil.

*By a thin foil is meant one where there is little modification of the neutron distribution due to the presence of the foil, i.e., one for which $\sigma_{tot} n \delta \ll 1$, where σ_{tot} is the total cross section, n the number of atoms per cm³, and δ the foil thickness.

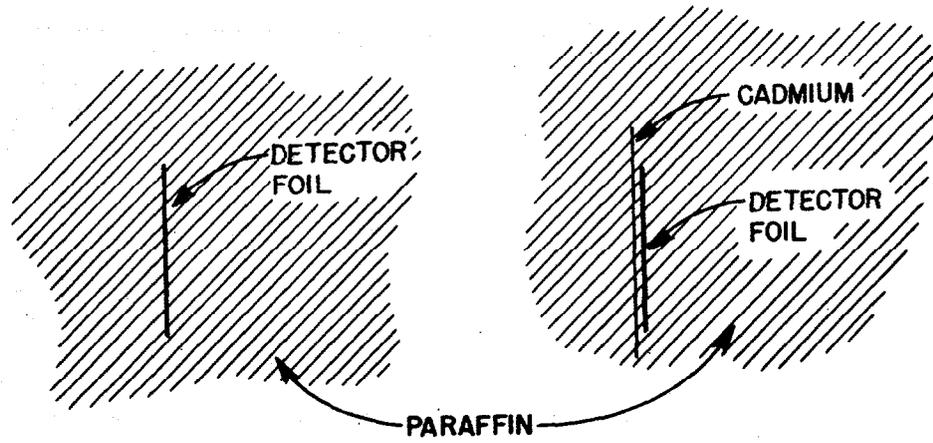


Figure 46. Measurement of albedo.

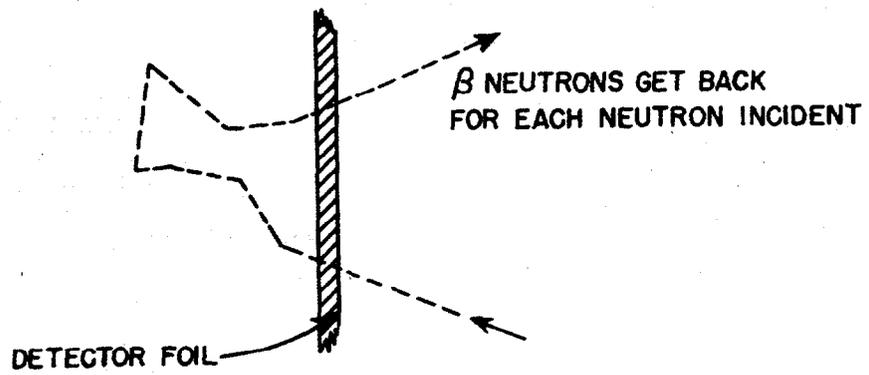


Figure 47. Multiple transmission in albedo experiment.

a-93-8

PROBLEMS

1. The diffusion length for thermal neutrons in water is 2.8 cm. Now the neutron distribution is Maxwellian, so the average velocity is $V = (2/\sqrt{\pi}) v$ (where v is the velocity in $mv^2/2 = kT$) or $V = \sqrt{8kT/\pi m}$. To calculate the mean free path for absorption (Λ), we must use σ_a for neutrons with velocity V rather than v . Given $\sigma_a = 0.31$ barns for room temperature neutrons ($v = 2200$ m/sec; see Figure 11, page 24), calculate Λ for V neutrons. What is the transport mean free path for thermal neutrons (Maxwell distribution) in water?

2. If ξ is the average fraction (averaged over angles of incidence) of incident neutrons absorbed in the foil, find the relation between Λ/B and the albedo.

CHAPTER VIII

NUCLEAR FISSION

8.1 THE BINDING ENERGIES OF NUCLEI

Before we proceed to take up the subject of fission, it is necessary to understand the general nature of the forces that hold nuclei together. For this purpose, we should like to find an expression for the nuclear binding energy. As we have seen in Chapter III the nuclear mass is related to the binding energy. The relation is simply:

$$M = (A - Z)M_n + ZM_p - (\text{Binding energy} / c^2) \quad (8-1)$$

with M the nuclear mass, M_n the neutron mass, M_p the proton mass, and A, Z the mass and atomic numbers, respectively. This relationship shows that we may check any conclusions about nuclear binding energies by comparison with nuclear masses.

In the absence of exact knowledge concerning the nuclear forces, the problem of finding the dependence of binding energy on Z and A is a difficult one. We must examine our empirical knowledge about nuclei for implications concerning the nuclear forces of the binding energy. Our empirical knowledge includes:

- Nuclear size and constancy of density of nuclear matter.
- Tendency of Z to be equal to $A/2$.
- Effectiveness of Coulomb forces in making Z less than $A/2$.
- Rarity of nuclei with even A and odd Z .

Each of these factors will be considered separately in its effect on the nuclear binding energy.

Consider first the nuclear size. From scattering and other experiments with heavy nuclei it is found that nuclear radii are proportional to $A^{1/3}$. In fact,

$$R = 1.48 \times 10^{-13} A^{1/3} \text{ cm} \quad (8-2)$$

fairly well fits the known data, although this formula does not mean much if applied to the very lightest nuclei. For the present purpose, the formula implies that the average density of constituent particles is about the same in all nuclei. It is quite likely that the density within a single nucleus does not vary much from one region within the nucleus to another. If a certain binding energy resulting from nuclear forces is to be associated with two nuclear particles within the nucleus a given distance apart, it is clear from the foregoing that this binding energy per unit volume of the nucleus is constant, inasmuch as the average distances between constituent particles are everywhere the same. We conclude, therefore, that the binding energy of nuclei is essentially proportional to their volume or to A . In terms of the energy of the nucleus (the negative of the binding energy), we have then $E_1 \sim -a_1 A$, where a_1 is some positive coefficient which these considerations have not sufficed to determine. This is not completely accurate, since we have failed to consider the fact that the nuclear constituents at the nuclear surface are not bound as strongly as particles inside. The number of such particles is proportional to the surface area so that we must subtract a number proportional to $A^{2/3}$ (or R^2) from our previous estimate E_1 . Thus:

$$E_1 = -a_1 A + a_2 A^{2/3} \quad (8-3)$$

The second consideration, the tendency of Z to be equal to $A/2$, should be taken into account. Since A is the total number of nuclear particles (neutrons plus protons) this tendency means that there is a tendency for the number of protons to be the same as the number of neutrons. (While it is true that for heavy nuclei there are fewer protons than neutrons, we shall assume that this is due to the electrostatic repulsion between protons, which will be considered next. In other words, we are assuming that if it were not for Coulomb forces between protons, there would be equal numbers of protons and neutrons in nuclei.) There are at least three types of nuclear forces within a nucleus: neutron-proton, proton-proton, and neutron-neutron. In view of the equality of the number of protons and neutrons in nuclei, the last two types of forces must be of the same order of magnitude. For if the proton-proton forces were stronger, nuclei with more protons than neutrons would tend to be more strongly bound, hence more stable than those with equal numbers of each. If the energy of isobars (same A , different Z) were plotted against Z , we should get a curve symmetric about $Z = A/2$, as nuclei with Z protons and $A-Z$ neutrons would have the same energy as those with Z neutrons and $(A-Z)$ protons, assuming equality of neutron-neutron and proton-proton forces. The isobar curve as shown in Figure 48 shows a minimum at $Z = A/2$ since nuclei for which $Z = A/2$ are the most stable. Thus E_2 , the energy associated with the departure from equality in the number of protons and neutrons, must be proportional to some even power of $Z - (A/2)$. For simplicity, consider that in the neighborhood of $Z = A/2$, the energy E_2 is proportional to the square of $Z - (A/2)$. To see what the "dimensions" of the coefficient should be, consider two nuclei with the same value for Z/A , one having an A twice the other, so that both nuclei have

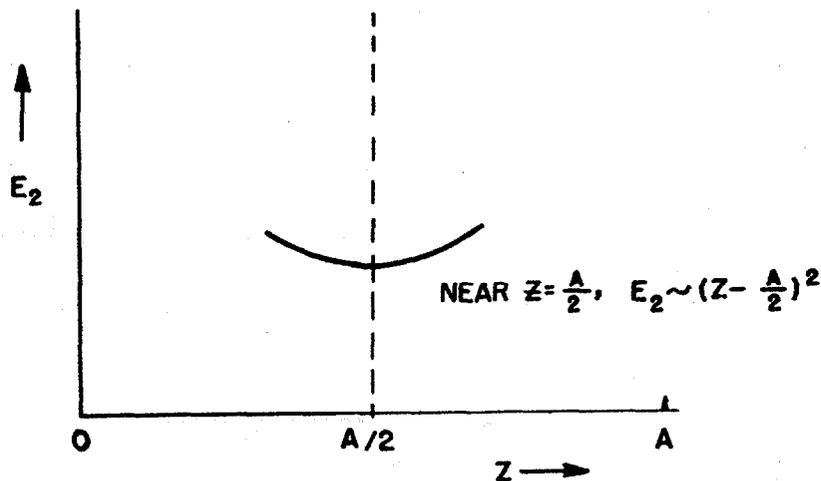


Figure 48. Quadratic shape of energy surface.

the same fractional excess of neutrons over protons, but one has twice as many particles. The larger nucleus will have twice the E_2 if we associate with each extra or unpaired particle a certain fixed energy. It follows then that E_2 should be proportional to A , or:

$$E_2 = a_3 A \left(\frac{Z-1}{A} \right)^2 = a_3 \frac{(A-Z)^2}{A} \quad (8-4)$$

The effect of Coulomb forces between protons can be taken into account by finding the energy E_3 associated with the Z protons being distributed in a sphere of radius R . This is a simple problem in electrostatics if we consider the charge uniformly distributed throughout the sphere. In this case, the potential energy of a charge Ze uniformly distributed throughout a sphere of radius R is just $(3/5)(Ze)^2/R$ in ergs (e in esu). Substituting for R from equation (8-1) and for " e " and then converting ergs to mass units, the Coulomb energy becomes:

$$E_3 = 0.000627 Z^2/A^{1/3} \quad (8-5)$$

Our last consideration concerns the dependence of the binding energy on the even or odd numbers of protons and neutrons. It has been found empirically that there are few stable nuclei with even atomic weight A and odd atomic number Z . In fact, it can be said that the most stable nuclei tend to have both Z and $A-Z$ even. Slightly less stability occurs in the cases Z odd, $A-Z$ even, and Z even, $A-Z$ odd. Clearly forces between nuclear constituents must, therefore, show a dependence on whether an even or odd number of neutrons and protons are about and so must the binding energy. An explanation has been advanced based on the idea that constituents tend to fill the nucleus' lowest energy levels and that strong forces exist between the pairs of neutrons or protons that can fill the same level. It has been empirically determined that $E_4 = \delta$ can be assigned as a correction term to our expression for the binding energy on the following basis (δ in mass units):

$$\begin{aligned} \delta &= 0 \text{ for } A \text{ odd} \\ \delta &= -0.036/A^{3/4} \text{ for } A \text{ even, } Z \text{ even} \\ \delta &= +0.036/A^{3/4} \text{ for } A \text{ even, } Z \text{ odd} \end{aligned} \quad (8-6)$$

Combining the various terms, equations (8-3) to (8-6) and substituting in equation (8-1), the nuclear mass (in mass units):

$$M = 1.00893 A - 0.00081 Z - a_1 A + a_2 A^{2/3} + a_3 \frac{(A-Z)^2}{A} + 0.000627 \frac{Z^2}{A^{1/3}} + \delta$$

We must now evaluate the coefficients a_1 , a_2 , and a_3 . First, a_3 is evaluated by setting $dM/dZ = 0$. The resulting equation between Z and A ,

$$Z_A = \frac{0.00081 + a_3}{2a_3 + 0.001254A^{2/3}} \cdot A \quad (8-7)$$

is one for which M is a minimum and, therefore, gives the stablest values of Z for any A . Fitting this equation to the known stable isotopes gives a best value for a_3 of 0.083 in mass units. The other constants, a_1 and a_2 , are determined by fitting the equation for M to the known data for nuclear masses, with the resulting values $a_1 = 0.00504$ and $a_2 = 0.014$. Hence the complete expression for the nuclear mass as a function of A and Z is:

$$M(A,Z) = 0.99389 A - 0.00081Z + 0.014A^{2/3} + 0.083 \frac{(A-Z)^2}{A} + 0.000627 \frac{Z^2}{A^{1/3}} + \delta \quad (8-8)$$

a-93-89

This formula can be used for calculation of the binding energies of neutrons to isotopes of uranium. This information will be very closely connected with the ability of slow neutrons to fission these various isotopes, as we shall see. Let us calculate the binding energy of a neutron to U^{235} .

$$\begin{aligned} U^{235} : M &= 235.11240 \text{ [from equation (8-8)]} \\ \text{Neutron: } M &= \underline{1.00893} \\ \text{Sum} &= 236.12133 \\ U^{236} : -M &= \underline{-236.11401} \text{ [also equation (8-8)]} \end{aligned}$$

$$\text{Binding energy} = 0.00732 \text{ mass units or } 6.81 \text{ Mev}$$

Similarly the binding energies of neutrons to U^{236} , U^{237} , and U^{238} would be 5.51, 6.56, and 5.31 Mev, respectively. The alternation of the magnitudes of the binding energies comes from the factor δ . This alternation is superposed on the regular variation of $M(A,Z)$ with A and Z given by the other five terms of equation (8-8).

Additional examples of this type are given in the problems at the end of the chapter.

8.2 THE FISSION PROCESS - ENERGY CONSIDERATIONS

The packing fraction curve (see Section 3.2 and Figure 15) shows that in the region of uranium, the packing fraction is of the order 0.0006, whereas for middle-weight nuclei it is of the order -0.0007. This implies that the heavy nuclei are not energetically stable against breaking into two middle-sized nuclei. Examining this more closely, we see that the energy that would be released in such a splitting is $M(A,Z) - 2M(A/2, Z/2)$. If this is positive, the splitting is energetically possible. This difference can be written in terms of the packing fractions:

$$A \left[\frac{M(A,Z) - A}{A} - \frac{M\left(\frac{A}{2}, \frac{Z}{2}\right) - \frac{A}{2}}{\frac{A}{2}} \right] \quad (8-9)$$

or A times the difference in the packing fractions. Thus, when the difference between the packing fractions is positive, then fission is energetically possible. It is to be noted that the packing fraction difference does not give the energy released in a fission process. In Figure 49, the curve of N versus Z is shown (see also Figure 12). The transition from P to Q on the diagram results in an energy release proportional to the packing fraction difference. Actually in fission the end state is on the curve of stable isotopes at point R in Figure 49. Since R is at a lower mass point, the energy release in fission is *greater* than that given in equation (8-9). For example, if A is 240, then $A/2$ is 120; Z_A is 93.74, equation (8-7), for $A = 240$, and $Z_{A/2}$ is then 46.87. Using $A/2 = 120$ and the formula for Z_A gives the stable Z_{120} as 51.15, or about 4 units from $Z_{A/2}$. (That is $Z_{A/2} < Z_{A/2}$.) This means that about four beta particles will be emitted per fragment after fission.

From the packing fraction curves, it appears that fission is exoergic for all nuclei with A greater than 100. Why, then, is fission such a rare process? Consider a nucleus that breaks into two fission fragments. Plot the energy of the nucleus, i.e., the fragments, as a function of the distance between the two parts. At infinite separation, the energy is taken to be zero. When the fragments are combined ($r = 0$), we know from measurements that the energy is about 200 Mev or greater. What about points between $r = 0$ and infinity? Up to a distance of the order of the diameter of the fragments, it is the Coulomb energy between the particles alone that contributes to the energy between particles, since that is the only force acting between the fragments. This energy is $(Ze/2)^2/r$. When r is less than the diameter of the fragments, the energy must change in such a way that it becomes the fission energy (200 Mev) at $r = 0$. If $(Ze/2)^2/r$ is smaller than, equal to, or greater than the fission energy at $r =$ diameter of the fragments, then there are three corresponding transition curves, Figure 50, which can

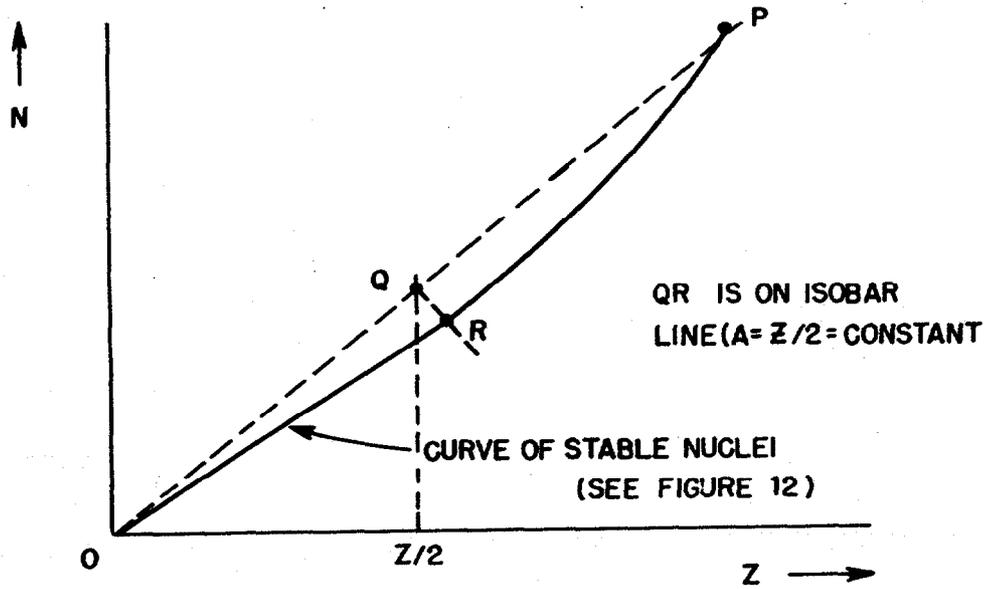


Figure 49. Energy release on fission.

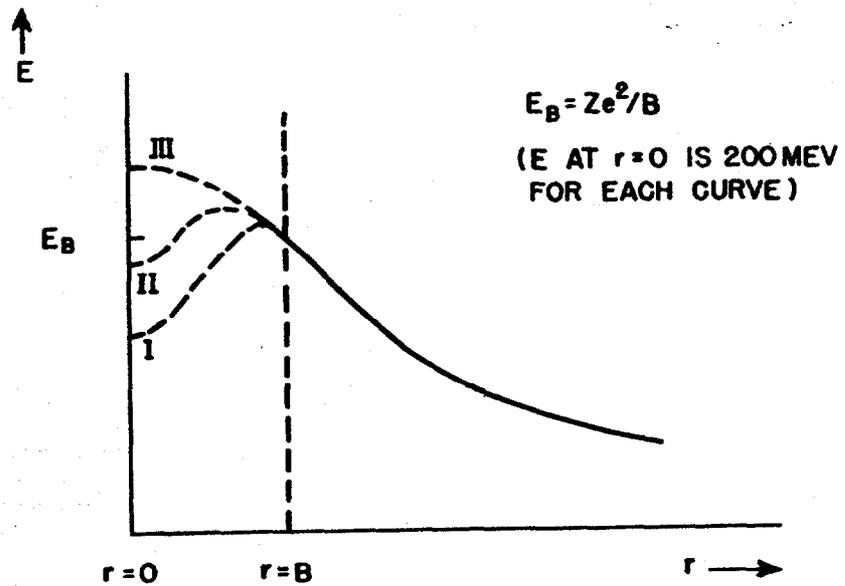


Figure 50. Transition curves for fission fragments.

a-93-9

be drawn to connect the Coulomb potential curve to the known energy at $r = 0$. Presumably stable nuclei with $A > 100$ are represented by curves of the type I, with barrier heights of the order of 50 Mev, since the Coulomb potential at $r =$ nuclear diameter is greater than the fission energy. Presumably uranium would be represented by a curve like II, where the barrier is about 6 Mev. Substances whose energy curve would be given by III would naturally not exist for long. This latter curve presumably represents nonexistent transuranics.*

Consider $r = B$ to be of the order of the diameter of a fission fragment. Then from equation (8-2), $B = 2 \times 1.48 \times 10^{-13}(A/2)^{1/3}$. Using this value for B , we can plot E_B (the Coulomb potential at $r = B$) as a function of mass number A , Figure 51. Similarly we can draw a curve E_A , the excess of mass of a parent nucleus of mass number A over that of its two fragments (i.e., the fission energy). This latter curve becomes negative below $A = 85$ and crosses the curve for E_B at about $A = 250$. From such a graph, one can get $E_B - E_A$ for any A . The quantity $E_B - E_A$ is a measure of the height of the energy barrier against fission.

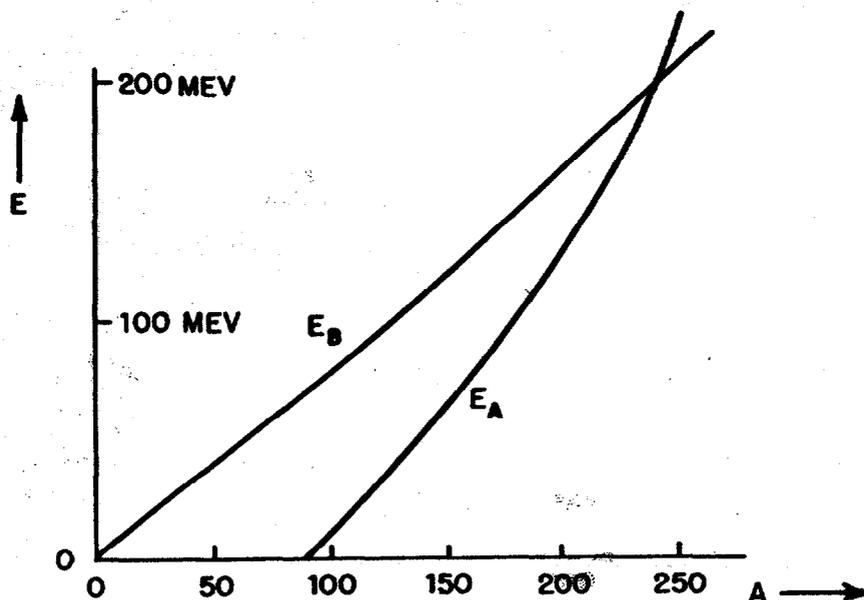


Figure 51. Energy barrier against fission.

* For a somewhat more detailed discussion of the transition curves, see the Bohr, Wheeler paper, Phys. Rev. 56:426 (1939).

It is possible, of course, to investigate more precisely the shape of the energy versus fragment separation curve near $r = 0$ if some specific model is assumed. Considering the Bohr liquid drop model, we assume that the original nucleus is a sphere and then calculate the change in energy for a small deformation. Let us further assume that the sphere, in beginning to split, deforms in a very simple manner, namely, it stretches slightly in one direction and flattens out perpendicularly to this direction, thus becoming an ellipsoid. If we assume that the sphere does not change its volume on becoming an ellipsoid, and this is reasonable in view of the fact that all nuclei tend to maintain the same density of nuclear particles, the change in the energy of the nucleus upon deformation will be due to only two of the five factors discussed in the last section. First, the *surface energy* will tend to *increase* with deformation because more surface will be exposed. Second, the *electrostatic energy* will *decrease* because the repelling charges will be effectively separated to some extent. Thus we have at least two energies changing in opposite ways with deformation of a spherical nucleus. The surface or capillary energy is proportional to the surface area or $A^{2/3}$, and the electrostatic energy is proportional to $Z^2 A^{1/3}$, which is $\sim A^{5/3}$. The latter energy becomes more important for heavy nuclei so that for heavy nuclei it is likely that the energy of a nucleus tends to decrease with deformation, making a spherical nucleus unstable. The opposite is true for light nuclei. From this picture, it is in heavy nuclei that we would expect fission.

Let us investigate in some detail the change of energy of a spherical nucleus upon distortion. The surface energy is proportional to the surface area, which for an ellipsoid is:

$$2\pi b^2 + 2\pi \frac{a^2 b}{\sqrt{a^2 - b^2}} \arccos(b/a)$$

a and b are the semimajor and semiminor axes, respectively. (Note that for $b = a$ this reduces to $4\pi b^2$, since $\arccos(b/a) \approx \arctan(\sqrt{a^2 - b^2}/b) \approx \sqrt{a^2 - b^2}/b$.) The electrostatic energy of a charge distributed throughout the volume of an ellipsoid can be shown to be:

$$\frac{3}{10} \frac{Z^2 e^2}{\sqrt{a^2 - b^2}} \log_e \frac{a + \sqrt{a^2 - b^2}}{a - \sqrt{a^2 - b^2}}$$

(This reduces to $(3/5)(Z^2/r)$ for $b = a = r$.) Now consider a sphere of original radius R . If it is stretched in one direction, then $a = R(1 + \epsilon)$. The minor axis b changes so as to keep the volume of the sphere constant, i.e., $(4\pi/3)ab^2 = (4\pi/3)R^3$, from which $b = R/\sqrt{1 + \epsilon}$. Substituting for a and b in the two energy expressions above and developing the results in powers of ϵ , we find the electrostatic energy to be:

$$\frac{3}{5} \frac{Z^2 e^2}{R} \left[1 - \frac{1}{5} \epsilon^2 + \dots \right]$$

and the surface energy to be:

$$4\pi R^2 \left[1 + \frac{2}{5} \epsilon^2 + \dots \right]$$

It is to be noted that the first terms in each of these equations are simply the electrostatic energy and surface area, respectively, of the undistorted sphere, while the second terms are the corrections for distortion. The correction term for the electrostatic energy is negative, indicating a decrease with distortion; that for the surface area is positive, corresponding to an increase in area. Using the proper coefficients for these energies from the formula for $M(A, Z)$, equation (8-8), the excess in energy of the ellipsoid over the sphere is:

$$\epsilon^2 \frac{2}{5} \left[(0.014 A^{2/3}) - \frac{1}{5} (0.000627) \frac{Z^2}{A^{1/3}} \right]$$

The condition for stability against deformation is that the bracket be positive, that is:

$$\frac{Z^2}{A} < 44.7 \quad (8-10)$$

For uranium Z^2/A is 36, while for lower elements its value is even smaller, indicating that the foregoing condition for instability is too stringent.

Actually, instability will occur for lower values of Z^2/A than the limiting value given by equation (8-10). For potential curves of the shape shown in Figure 52, the barrier is rather transparent, and one could expect appreciable spontaneous fission. The probability of leakage through the barrier will be finite, so that the decay constant with respect to fission will not be zero. Even for U^{238} , there are ~ 20 fissions per gram per hour spontaneously (corresponding to a "fission half-life" of about 10^{16} years), so that for heavier atoms this may soon become a prominent phenomenon.

Our principal interest is not in spontaneous fissions but in fissions brought about by neutrons. Neutrons can cause fissions by contributing their kinetic energy and their binding energy to the nucleus. This energy is at least 5 or 6 Mev (the binding energy of the neutron) and may raise the energy of the nucleus high enough within the barrier for a fission to take place before the excess energy is lost by gamma radiation. Because of the fact that the binding energy of neutrons to nuclei with an odd number of neutrons is larger than it is to those with an even number of neutrons (see equation (8-6)) it is reasonable to expect fission for thermal neutrons to be more prevalent for those nuclei with an odd number of neutrons. This is confirmed by experiment. U^{238} is not fissioned by thermal neutrons whereas U^{235} is. Moreover the other "fissionable" materials, U^{233} and Pu^{239} , each have an odd number of neutrons. From facts such as these and photofission thresholds, one can estimate that for uranium the height of the fission barrier is of the order of 5 Mev.

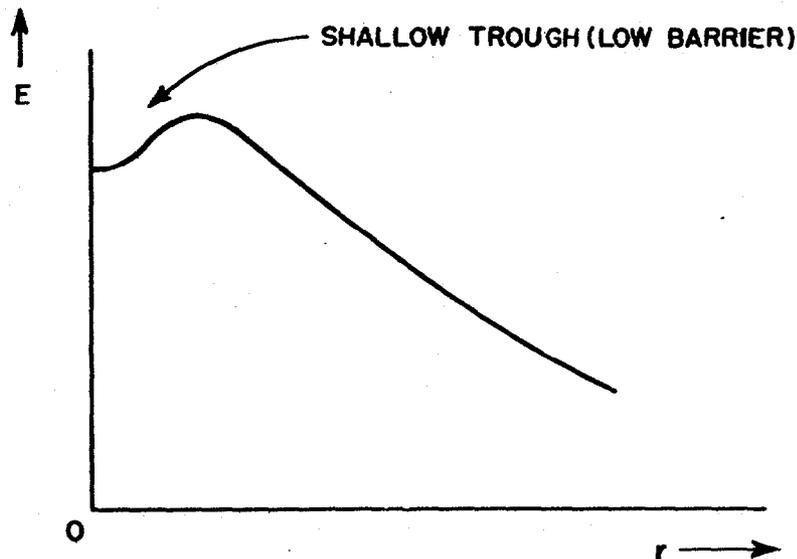


Figure 52. Possible potential barrier in spontaneous fission.

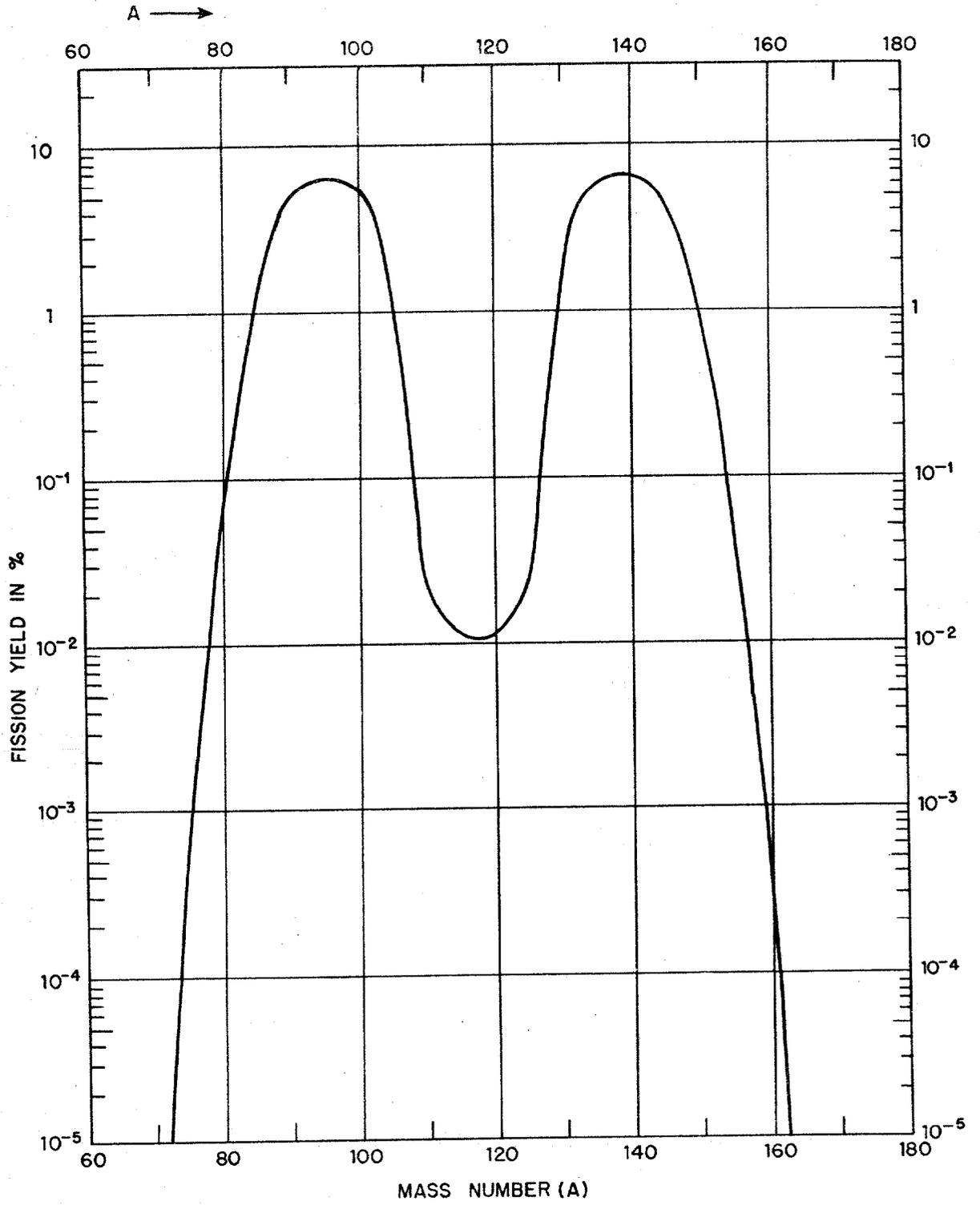


Figure 53. Fission product yield for U 235.

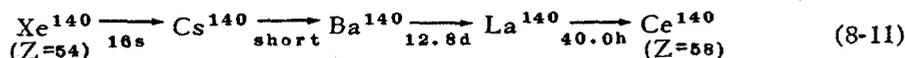
A-93-95

It is to be kept in mind that in the consideration of the competition of fission with other processes, it is not sufficient to consider energies alone, as we have done. For fission, one must not only have the energy rise to the top of the barrier, but it is also necessary that this energy be concentrated in the proper modes of motion for fission. This may take some time, so that competing processes may occur at the expense of fission. Since the number of modes, and hence of useless nonfission modes, increases with excitation energy, it may, therefore, be very likely that the reason photofission with ~ 100-Mev gamma rays on lower nuclei has not been observed is because the energy does not get concentrated in a proper mode before it is lost by some other way than fission.

8.3 THE FISSION PROCESS - RESULTING PARTICLES

When a nucleus fissions there are, in addition to the two fragments, a number of other particles observed. They are neutrons, beta particles, gamma rays, and often fast alpha particles. Some of these are observed to accompany the fission process, and others are emitted at various times following the fissioning itself.

The fission fragments themselves have been studied in great detail*. The distribution of fission fragments for U^{235} as a function of mass, shown in Figure 53, consists of two nearly identical peaks with maxima at mass numbers 96 and 140. If U^{235} is fissioned by a neutron, and in the course of fissioning two neutrons are emitted, then the mass number corresponding to equal splitting would be $A = 117$. The observed yield for equal splitting ($A = 117$) is only 0.01%, whereas the maximum yields (at $A = 96$ and 140) are about 6.5%. In each case, the fission fragment formed is unstable (see Figure 49) because of the excess of neutrons. For example, the nucleus with mass number 140, yield 6.3%, finishes up as stable Ce^{140} after a sequence of beta decays:



A large number of such "fission product chains" have been identified.

It should be noted that the distribution of masses also gives the distribution of the relative kinetic energies of fission fragments. This follows from the conservation of momentum. If E_1 and E_2 are the kinetic energies of the fission fragments M_1 and M_2 , then conservation of momentum requires that $M_1 V_1 = M_2 V_2$ or $\sqrt{2M_1 E_1} = \sqrt{2M_2 E_2}$. Thus $M_1/M_2 = \sqrt{E_2/E_1}$, so that the heavier of a pair of fission fragments has the smaller kinetic energy. The absolute value of the energy E_1 corresponding to the mass M_1 follows from the fact that the total energy $E_1 + E_2$ is a constant (~ 160 Mev) and the total mass $M_1 + M_2$ is a constant (~ 234 m.u.).

The neutrons emitted in fission are classed as either "prompt" or "delayed". The term "prompt" means that the neutrons leave the fission fragment after its formation in times shorter than we can measure. An estimate of 10^{-15} second can be made by considering the fission product as the splitting of a drop. The final fragments are not of spherical shape (Figure 54), so there will be a considerable vibrational energy associated with oscillations about the equilibrium (spherical) shape of the fragment. This excitation energy may be sufficient to evaporate a neutron, especially since neutron binding energies in fission fragments are small because of the excess of neutrons. For example, assume that U^{235} is made to fission by a neutron, and two fragments with $A = 118$ and $Z = 46$ appear. Using the formula for $M(A, Z)$, equation (8-8), one can calculate binding energies for various nuclei of weight $A = 118$:

*See "Nuclei Formed in Fission: Decay Characteristics, Fission Yields and Chain Relationships" issued by the Plutonium Project in J. Am. Chem. Soc. 68: 2411(1946) and Rev. Mod. Phys. 18: 513(1946).

9-93-96

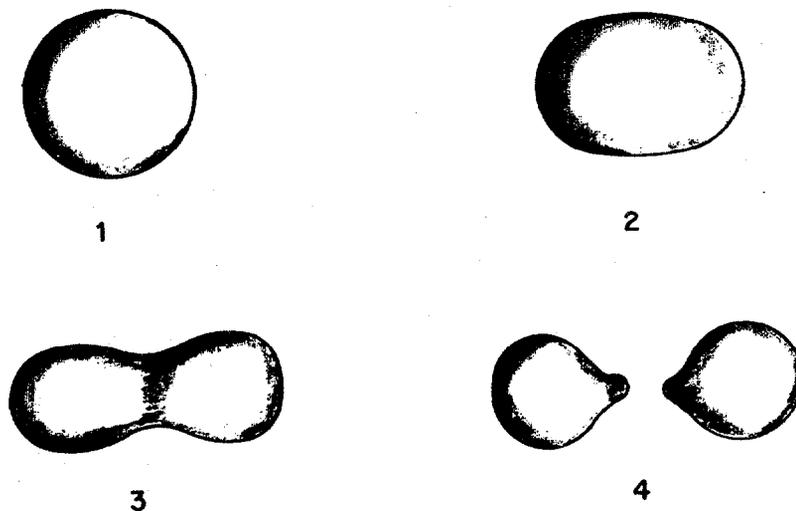


Figure 54. Fission of a liquid drop.

Z	44	45	46	47	48	49	50
Neutron Binding Energy (Mev)	2.5	6.8	3.6	7.8	4.7	9.0	5.8

(where $Z = 50$ is the stable value for Z if $A = 118$). Thus, neutrons may be lightly bound to fission fragments. Whenever neutron emission is energetically possible, neutron emission is likely, because of the absence of a barrier for neutrons. As a matter of fact, one could conclude from observations that considerable excitation energy must be present in the fragments, because from one to three neutrons are emitted per fission in the case of U^{235} .

The energies of the neutrons that come off at fission are given in the distribution curve in Figure 55. In the center of gravity system of neutron and fission fragment, the neutron energy distribution would be approximately Maxwellian, with a "temperature" corresponding to the excitation of the fragment. To get the theoretical curve for the distribution in the laboratory system, one would have to take account of the motion of the fission fragment and the dependence of emission probability on neutron energy.

In addition to these prompt neutrons about 1% are delayed. To explain the emission of delayed neutrons, consider (Figure 56) a fragment A, which undergoes a β disintegration to a nucleus B. Usually this disintegration will go to the ground state of B, but occasionally the nucleus B may end up in an excited state with excitation energy greater than B. E., the binding energy of a neutron. In such a case, neutron emission becomes quite likely. Such neutrons would come off very

A-93-1

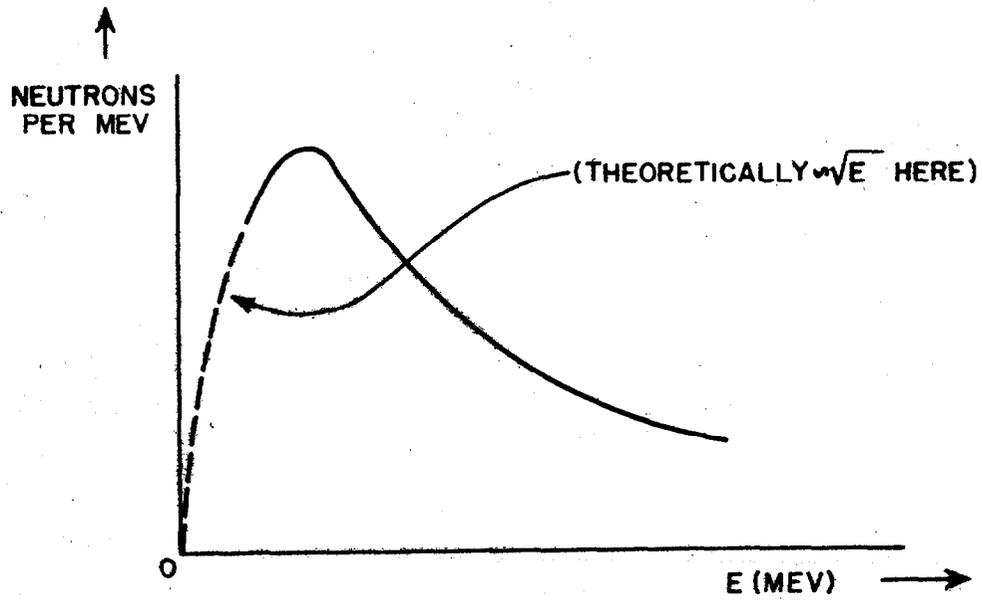


Figure 55. Prompt neutrons per unit energy versus energy.

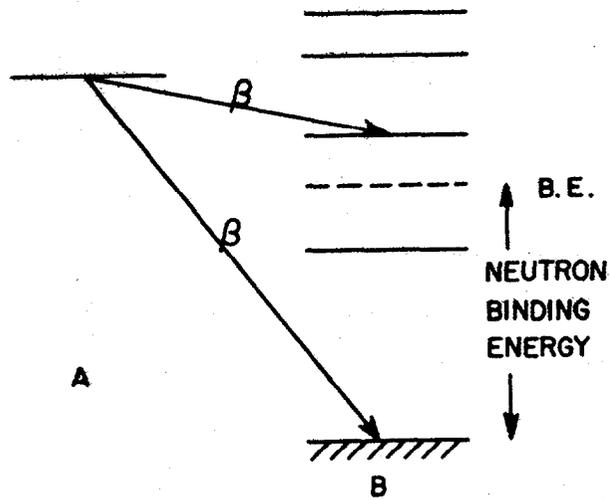


Figure 56. Mechanism of delayed neutron emission.

quickly after the β decay and would, therefore, show decay periods that correspond to the periods of the disintegration of A to the excited state of B.

The delayed neutron periods that have been observed and their yields are listed in Table 4. The first two periods are rather well verified and have been identified to be emitted by Kr^{87} and Xe^{137} , respectively*. The shortest period is not yet definitely confirmed.†

TABLE 4 - DELAYED NEUTRONS FROM U^{238} [Phys. Rev. 74:1330(1948)] †.

HALF-LIFE (SEC)	RELATIVE INTENSITY	DELAYED NEUTRONS PER PROMPT NEUTRON. (%)
55.3 ± 0.7	0.054	0.025
22.4 ± 0.4	0.294	0.166
5.5 ± 0.3	0.297	0.213
1.7 ± 0.2	0.279	0.241
0.36 ± 0.07	0.076	0.085
$(0.0044 \pm 0.0007 ?)$	(~ 0.02)	

8.4 THE FISSION PROCESS - GENERAL NATURE OF CROSS SECTION

The cross sections for capture and fission for the isotopes of uranium and plutonium that have so far been investigated show a rather complicated dependence on energy. For some isotopes, the (n,f) cross section decreases with neutron energy, whereas for others it increases. Some isotopes show an (n,f) threshold, whereas some have an (n,f) cross section that follows the $1/v$ law at low energies. Pronounced resonances for capture are apparent at low neutron energies in isotopes like U^{238} . The resonances become less striking at higher energies, where they tend to become smeared out.

*Phys. Rev. 72:545 (1947).

†Phys. Rev. 74:1330 (1948).

†Similar periods have been found for the delayed neutrons in plutonium fission. See Phys. Rev. 72:567 and 570 (1947). The per cent of delayed neutrons per prompt neutron is taken from Phys. Rev. 73:111 (1948).

A-93-94

PROBLEMS

1. Substituting the correct value for a_3 into equation (8-7) gives a relation between Z_A and A . Compare points predicted by this equation with the corresponding values for about ten known stable isotopes. Make the comparison graphical.
2. Dempster, in *Physical Review* 53:870 (1938), gives a curve of the packing fraction vs. the mass number. Using equation (8-8), plot packing fractions expected "theoretically" along with Dempster's experimental curve and note the degree of agreement.
3. Calculate the binding energies of neutrons to Th^{232} , Au^{197} , Sm^{149} , In^{115} , and Mn^{55} . Use the formula for $M(A, Z)$, as was done at the end of Section 8.1. Sufficiently accurate *experimental* data for nuclear masses (for such an application as this) exist only for the lightest nuclei.
4. At what atomic number is instability reached according to the inequality of equation (8-10)? Use the expression for Z_A , the proper value of Z for a nucleus of weight A , that has been developed in equation (8-7).
5. Derive an energy distribution curve for fission neutrons, assuming a velocity V for the fission fragment and a Maxwell distribution of energies in the center of gravity system. Also assume that the probability of neutron escape is proportional to their velocity.